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Transparent boundary conditions for the Helmholtz equation in some ramified domains with a fractal boundary

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Abstract

The paper addresses a class of boundary value problems in some self-similar ramified domains, with the Laplace or Helmholtz equations. Much stress is placed on transparent boundary conditions which allow the solutions to be computed in subdomains. A self similar finite element method is proposed and tested. It can be used for numerically computing the spectrum of the Laplace operator with Neumann boundary conditions, as well as the eigenmodes. The eigenmodes are normalized by means of a perturbation method and the spectral decomposition of a compactly supported function is carried out. Finally, a numerical method for the wave equation is addressed.

Keywords: Helmholtz equation; Transparent boundary conditions; Domains with fractal boundaries

1. Introduction

This paper is concerned with numerical methods for some boundary value problems in a self-similar ramified domain of $\mathbb{R}^2$ with a fractal boundary. This work was originally inspired by a wider and challenging project aimed at simulating the diffusion of medical sprays in lungs, see [10,23,24] for accurate physical descriptions of the lungs' physiology and for studies concerning the diffusion of oxygen in lungs. Our goals are much more modest, since the geometry is quite simpler (only two dimensions) and since we restrict ourselves to the Laplace and Helmholtz equations. Yet, we hope that rigorous results and methods will prove useful. Other applications can be found, for example in chemical engineering, see [8].

The geometry under consideration is displayed in Fig. 1. The domain Ω^0 is constructed in an infinite number of steps, starting from a simple polygonal T-shaped domain of $\mathbb{R}^2$, called Y^0 below; we call Y^n the domain

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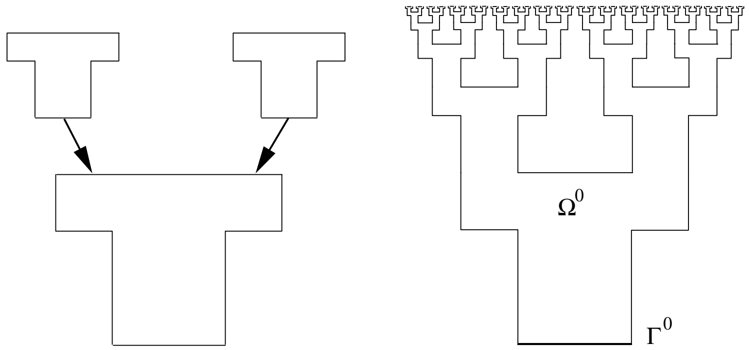


Fig. 1. Left: the first step of the construction: two dilated/translated copies of Y^0 are glued to Y^0. Right: the infinitely ramified domain Ω^0 (only a few generations are displayed).

obtained at step n: Y^{n+1} is obtained by attaching Y^n to 2^{n+1} dilated/translated copies of Y^0, with the dilation factor of $1/2^{n+1}$. So we have $Y^0 \subset Y^1 \subset \cdots \subset \Omega^0$. We say that Ω^0 is self similar, because $\Omega^0 \setminus \overline{Y^n}$ is made out of 2^{n+1} dilated copies of Ω^0 with the dilation factor of $1/2^{n+1}$. The boundary of Ω^0 is made up of three parts; two straight lines, the bottom (resp. top) boundary Γ^0 (resp. Γ^∞) of Ω^0, and the lateral part of the boundary Σ^0.

The present work is devoted to Poisson problems in Ω^0 with the Laplace or Helmholtz equations, Dirichlet conditions on Γ^0 and homogeneous Neumann conditions on the remaining part of the boundary. For example, the Helmholtz equation is used in a simplified model for time harmonic acoustic waves: we use the restrictive term *simplified* to keep in mind that this equation may be inappropriate when the length scales in the geometry become much smaller than the wavelength, since *viscothermal layer* effects become preponderant. The Laplace equation arises in e.g. electrostatics for computing the electrical potential, or in the simplest fluid models (potential flow). The content of this paper can be applied to more involved fluid models, e.g. Stokes equations, but we decided to focus on simple equations to stress the general ideas. Similarly, what follows applies to the equation $\operatorname{div}(\chi \operatorname{grad} w) = 0$, where χ is a symmetric positive definite constant tensor.

Partial differential equations in domain with fractal boundaries or fractal interfaces is a relatively new topic: variational techniques have been developed, involving new results on functional analysis, see [5,27,19,20]. A nice theory on variational problems in fractal media is given in [26].

The present work is the continuation of the more theoretical paper [1]: here we focus on numerical methods and possible applications. For the methods to be understood, we restate several results contained in [1], and some of the proofs when we think that they are necessary for a good understanding. All the missing proofs can be found in [1]. Note that in the last article and in [2], we also addressed nonhomogeneous Neumann conditions on Γ^∞, which appear relevant for modelling the lungs.

Much stress will be placed on a method for computing the restriction of the solution to the subdomains Y^n, with n fixed. This is important, because, in numerical simulations, it is not possible to completely represent the domain Ω^0, for this would imply an infinite memory and computing time. We shall show that it is possible to compute the solution in Y^n by successively solving $1 + 2 + \cdots + 2^n$ boundary value problems in the elementary domain Y^0, with what we call *transparent boundary conditions* on the top part of the boundary of Y^0, see Algorithms 1 and 3. Such boundary conditions involve nonlocal operators, which may be called Dirichlet–Neumann operators. These operators will be computed, more precisely approximated up to an arbitrary accuracy, by taking advantage of the self-similarity in the geometry. For Laplace's equation, the Dirichlet–Neumann operator is approximated as the limit of an inductive sequence, see Theorem 3. For the Helmholtz equation, the Dirichlet–Neumann operators (depending on the pulsation of the related harmonic wave), can be approximated by performing iterations of a renormalization operator, see Section 5.3.

The method is reminiscent of some of the techniques involved in the theoretical analysis of finitely ramified fractals, see [29,32,31], also [18,4,35] for review texts on analysis and probability on finitely ramified fractals and [28,11] for numerical simulations (note that in the present work, what is fractal is $\partial\Omega^0$, not Ω^0).

We shall carry out the whole program at the discrete level with finite elements and self-similar triangulations. The result is a method for computing the restriction of the solution to Y^n, n fixed: in the proposed discrete method, there are two sources of errors, the error due to the discretization and the error due to the approximation of the discrete transparent boundary condition; this last error can be made as small as desired (assuming infinite arithmetic precision), whereas the first error decays when the triangulation is refined.

The above mentioned Dirichlet–Neumann operators can be used for numerically computing the spectrum and the eigenmodes associated to a Neumann problem. This topic has been discussed in [34,33,12,21,22] where first numerical results were given on eigenfunctions on Koch flake domains, see also [7,9] in the context of finitely ramified fractals. In that respect, we believe that the method presented here will be useful since it takes into account the fine scales in the ramifications.

In order to solve time dependent problems in the irregular domain, the spectral information can be used, but for that, one needs to normalize the eigenmodes: in this paper, we propose a perturbation method for normalizing the eigenmodes. This allows the spectral decomposition of any function compactly supported in the domain to be found, and finally time dependent equations like the wave equation to be addressed.

The paper is organized as follows: in Section 2, the geometry is presented in detail. In Section 3, we briefly review some theoretical results on Sobolev spaces. Section 4 addresses the boundary value problems with Laplace's equation, as well as the transparent boundary conditions and the approximations of the Dirichlet–Neumann operator, at the continuous level. In Section 5, a similar program is carried out for the Helmholtz equation. The discrete analogues of the methods are described in Section 6, where a self-similar finite element method is proposed. Numerical results are presented in Section 7. Section 8 is concerned with two applications: first, the computation of the spectrum and of the eigenmodes, and their normalization by a perturbation technique; second, a method for solving a time-dependent problem (here the wave equation) from the knowledge of the spectral information.

The content of the paper can be generalized to other geometries: for example, straightforward generalizations are discussed in [2, Section 9]. It is also straightforward to extend all the methods presented below to the snowflakes domains discussed in [5] (note that the Hausdorff dimension of the boundary is greater than one in that case). In the forthcoming paper [3], we discuss the case when the rotation angles of the similarities used for constructing Ω^0 are nonzero, in which the Hausdorff dimension of Γ^∞ differs from one. In this case, some modifications are needed when dealing with nonhomogeneous Neumann conditions.

2. The geometry of the model problem

2.1. The domain Ω^0

Consider the following T-shaped subset of $\mathbb{R}^2$:

$$Y^0 = \text{Interior}(([-1,1]\times[0,2])\cup([-2,2]\times[2,3])).$$

Let F_1 and F_2 be the affine maps in $\mathbb{R}^2$

$$F_1(x) = \left(-\frac{3}{2}+\frac{x_1}{2}, 3+\frac{x_2}{2}\right), \qquad F_2(x) = \left(\frac{3}{2}+\frac{x_1}{2}, 3+\frac{x_2}{2}\right). \tag{1}$$

Note that F_1 is the homothety of ratio $\frac{1}{2}$ and center $(-3,6)$ and F_2 is the homothety of ratio $\frac{1}{2}$ and center $(3,6)$.

For an integer $n \geqslant 1$, we call $\mathscr{A}_n$ the set containing all the maps from $\{1,\ldots,n\}$ to $\{1,2\}$ (note that the cardinality of $\mathscr{A}_n$ is 2^n) and for $\sigma\in\mathscr{A}_n$, we define the affine map in $\mathbb{R}^2$

$$\mathscr{M}_\sigma(F_1,F_2) = F_{\sigma(1)}\circ\cdots\circ F_{\sigma(n)}. \tag{2}$$

Let us agree that $\mathscr{A}_0=\{0\}$ and that $\mathscr{M}_0(F_1,F_2)$ is the identity. The open domain Ω^0 is constructed as an infinite union of subsets of $\mathbb{R}^2$ obtained by translating/dilating Y^0

$$\Omega^0 = \text{Interior}\left(\bigcup_{n=0}^{\infty}\bigcup_{\sigma\in\mathscr{A}_n}\mathscr{M}_\sigma(F_1,F_2)(\overline{Y^0})\right). \tag{3}$$

The construction of Ω^0 is displayed in Fig. 1. It may be seen that $\Omega^0\subset(-3,3)\times(0,6)$.

Remark 1. Note that similar construction may be done using dilations with ratii α^n, $n \in \mathbb{N}$, with $\alpha \in (0, 1/2]$; here we have chosen $\alpha = 1/2$.

It is important to observe that, for the two points $A = (-3/2, 5/2)$ and $B = (3/2, 5/2)$, and call $A_n = (F_1^n \circ F_2)(A)$, $B_n = (F_2^n \circ F_1)(B)$, we have that

- $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = (0, 6)$, therefore $\lim_{n\to\infty} |A_n B_n| = 0$,
- $A_n \in \Omega^0$ and $B_n \in \Omega^0$,
- The length of any curve joining A_n and B_n that is contained in Ω^0 is greater than 3.

This implies that Ω^0 is not a (ϵ, δ) domain as defined in Jones [15] and also Jonsson and Wallin [16], or equivalently in dimension two a quasi-disk, see Maz'ja [25] so the general results for Sobolev spaces (see [15,16,25]) cannot be used.

2.2. The boundary of Ω^0

We define the bottom boundary of Ω^0 by $\Gamma^0 = ([-1, 1] \times \{0\})$ and $\Sigma^0 = \partial\Omega^0 \cap \{(x_1, x_2); x_1 \in \mathbb{R}, 0 < x_2 < 6\}$. Calling $\Gamma^\infty = [-3, 3] \times \{6\}$, one can check easily that

$$\partial\Omega^0 = \Gamma^0 \cup \Sigma^0 \cup \Gamma^\infty,$$

and that, in the last identity, the sets in the right hand side are disjoint.

2.3. Various subdomains of Ω^0

For what follows, it is important to define the polygonal open domain obtained by stopping the above construction at the step N, for $N \geqslant 0$

$$Y^N = \text{Interior}\left(\bigcup_{n=0}^{N} \bigcup_{\sigma\in\mathcal{A}_n} \mathcal{M}_\sigma(F_1, F_2)(\overline{Y^0})\right). \tag{4}$$

It will be useful to define the infinitely ramified set Ω^N

$$\Omega^N = \Omega^0 \setminus \overline{Y^{N-1}} = \text{Interior}\left(\bigcup_{n=N}^{\infty} \bigcup_{\sigma\in\mathcal{A}_n} \mathcal{M}_\sigma(F_1, F_2)(\overline{Y^0})\right). \tag{5}$$

The following self-similarity property is true: Ω^N is the union of 2^N nonoverlapping translated copies of $\frac{1}{2^N} \cdot \Omega^0$, i.e.

$$\Omega^N = \bigcup_{\sigma\in\mathcal{A}_N} \Omega^\sigma, \tag{6}$$

where

$$\Omega^\sigma = \mathcal{M}_\sigma(F_1, F_2)(\Omega^0). \tag{7}$$

The bottom boundary of Ω^N is defined by

$$\Gamma^N = \bigcup_{\sigma\in\mathcal{A}_N} \Gamma^\sigma \subset \left\{x : x_2 = 3\sum_{i=0}^{N-1} 2^{-i}\right\}, \tag{8}$$

where

$$\Gamma^\sigma = \mathcal{M}_\sigma(F_1, F_2)(\Gamma^0). \tag{9}$$

Let us stress that Γ^N is *strictly* contained in $\partial Y^{N-1} \cap \left\{x : x_2 = 3\sum_{i=0}^{N-1} 2^{-i}\right\}$.

3. Some function spaces on Ω^0

We are interested in boundary value problems in Ω^0. Since the boundary of Ω^0 is very irregular, a good way to give a sense to these problems (especially to boundary condition on Γ^∞) is to use variational or weak formulations. For that, we need to give some basic results on Sobolev spaces on Ω^0. For a more complete theory, including a result on traces of functions on Γ^∞, we refer to [1].

For $n \geqslant 0$ and $\sigma \in \mathscr{A}_n$, call $L^2(\Omega^\sigma)$ the space of square integrable function on Ω^σ. Consider the function space $H^1(\Omega^\sigma) = \{v \in L^2(\Omega^\sigma) \text{ s.t. } \nabla v \in (L^2(\Omega^\sigma))^2\}$, where ∇v is to be understood in the sense of distributions.

Of course, for all n, n', $0 \leqslant n \leqslant n'$, $\sigma \in \mathscr{A}_n$, $\eta \in \mathscr{A}_{n'}$ such that $\Gamma^\eta \subset \overline{\Omega^\sigma}$, it is possible to define the trace of $v \in H^1(\Omega^\sigma)$ on Γ^η. The trace operator on Γ^η is bounded from $H^1(\Omega^\sigma)$ to $L^2(\Gamma^\eta)$, and one can define the closed subspace of $H^1(\Omega^\sigma)$

$$\mathscr{V}(\Omega^\sigma) = \{v \in H^1(\Omega^\sigma) \text{ s.t. } v|_{\Gamma^\sigma} = 0\}. \tag{10}$$

We will use the notation $\lesssim$ to indicate that there may arise constants in the estimates, which are independent of the index n in Ω^n or Y^n, or the index σ in Ω^σ.

Theorem 1. *For all integer $n \geqslant 0$, and for all $\sigma \in \mathscr{A}_n$, for all $u \in H^1(\Omega^\sigma)$.*

$$\|u\|^2_{L^2(\Omega^\sigma)} \lesssim 2^{-2n}\|\nabla u\|^2_{L^2(\Omega^\sigma)} + 2^{-n}\|u|_{\Gamma^\sigma}\|^2_{L^2(\Gamma^\sigma)}. \tag{11}$$

For all $u \in H^1(\Omega^0)$,

$$\sum_{\sigma \in \mathscr{A}_n} \|u\|^2_{L^2(\Omega^\sigma)} \lesssim 2^{-n}\left(\|\nabla u\|^2_{L^2(\Omega^0)} + \|u|_{\Gamma^0}\|^2_{L^2(\Gamma^0)}\right). \tag{12}$$

Finally, the imbedding of $H^1(\Omega^0)$ in $L^2(\Omega^0)$ is compact.

4. A class of Poisson problems

4.1. Definition, existence and uniqueness results

Since the boundary of Ω^0 is very irregular, we are led to use variational or weak formulations (especially for boundary condition on Γ^∞).

Consider $u \in H^{\frac{1}{2}}(\Gamma^0)$. We are interested in the variational problem: find $w \in H^1(\Omega^0)$ such that

$$w|_{\Gamma^0} = u, \quad \int_{\Omega^0} \nabla w \cdot \nabla v = 0, \quad \forall v \in \mathscr{V}(\Omega^0). \tag{13}$$

It can be seen that (13) is a weak formulation of a Poisson problem, i.e. $\Delta w = 0$ in Ω^0, with a Dirichlet boundary condition on Γ^0, a homogeneous Neumann boundary condition on Σ^0, and a generalized homogeneous Neumann boundary condition on Γ^∞.

Proposition 1. *For $u \in H^{\frac{1}{2}}(\Gamma^0)$, problem (13) has a unique solution. This defines a bounded linear operator from $H^{\frac{1}{2}}(\Gamma^0)$ to $H^1(\Omega^0)$.*

4.2. The Dirichlet to Neumann operator and transparent boundary conditions

In fact, Proposition 1 can be extended by replacing Ω^0 with Ω^σ and Γ^0 by Γ^σ, for $\sigma \in \mathscr{A}_n$ and $n \geqslant 0$. This observation leads us to define the harmonic lifting operator $\mathscr{H}^\sigma$ from $H^{\frac{1}{2}}(\Gamma^\sigma)$ to $H^1(\Omega^\sigma)$: for all $u \in H^{\frac{1}{2}}(\Gamma^\sigma)$, the trace of $\mathscr{H}^\sigma(u)$ on Γ^σ is u and for all $v \in \mathscr{V}(\Omega^\sigma)$, $\int_{\Omega^\sigma} \nabla \mathscr{H}^\sigma(u) \cdot \nabla v = 0$. Since $\mathscr{A}_0 = \{0\}$, we denote by $\mathscr{H}^0$ the harmonic lifting in Ω^0. It is easy to verify that, for all $u \in H^{\frac{1}{2}}(\Gamma^\sigma)$,

$$\mathscr{H}^\sigma(u) \circ \mathscr{M}_\sigma(F_1, F_2) = \mathscr{H}^0(u \circ \mathscr{M}_\sigma(F_1, F_2)). \tag{14}$$

Theorem 2. *There exists a real number* ρ, $0<\rho<1$ *such that for all* $u\in H^{\frac{1}{2}}(\Gamma^0)$,

$$\int_{\Omega^N}|\nabla\mathscr{H}^0(u)|^2\leqslant\rho^N\int_{\Omega^0}|\nabla\mathscr{H}^0(u)|^2. \tag{15}$$

Remark 2. The estimate (15) is very important because it says that the contribution of Ω^N to the energy of $\mathscr{H}^0(u)$ decays exponentially as $N\to\infty$. This will allow the very accurate approximation of $\mathscr{H}^0(u)|_{Y^n}$ (n is a fixed number, for example $n=0$), by solving a boundary value problem in Y^n with nonlocal boundary conditions on Γ^σ, $\sigma\in\mathscr{A}_{n+1}$. It will be constructed by approximating the Dirichlet–Neumann operator T^σ on Γ^σ, which we introduce below.

For $\sigma\in\mathscr{A}_n$, one can define the operators T^σ, from $H^{\frac{1}{2}}(\Gamma^\sigma)$ to their respective duals by

$$\langle T^\sigma u,v\rangle=\int_{\Omega^\sigma}\nabla\mathscr{H}^\sigma(u)\cdot\nabla\mathscr{H}^\sigma(v)=\int_{\Omega^\sigma}\nabla\mathscr{H}^\sigma(u)\cdot\nabla\tilde{v} \tag{16}$$

for any function $\tilde{v}\in H^1(\Omega^\sigma)$ such that $\tilde{v}|_{\Gamma^\sigma}=v$. From the self-similarity of Ω^0, we have that

$$\forall u,v\in H^{\frac{1}{2}}(\Gamma^\sigma),\quad \langle T^\sigma u,v\rangle=\langle T^0(u\circ\mathscr{M}_\sigma(F_1,F_2)),(v\circ\mathscr{M}_\sigma(F_1,F_2))\rangle, \tag{17}$$

where the duality pairing in left (resp. right) hand side of (17) is the duality $(H^{\frac{1}{2}}(\Gamma^\sigma))'-H^{\frac{1}{2}}(\Gamma^\sigma)$ (resp. $(H^{\frac{1}{2}}(\Gamma^0))'-H^{\frac{1}{2}}(\Gamma^0)$).

Lemma 1. *For all* $u\in H^{\frac{1}{2}}(\Gamma^0)$, *for* $n\geqslant1$, *the restriction of* $\mathscr{H}^0(u)$ *to* Y^{n-1} *is the solution to the following boundary value problem*: *find* $\hat{u}\in H^1(Y^{n-1})$ *such that* $\hat{u}|_{\Gamma^0}=u$ *and* $\forall v\in\mathscr{V}(Y^{n-1})$,

$$\int_{Y^{n-1}}\nabla\hat{u}\cdot\nabla v+\sum_{\sigma\in\mathscr{A}_n}\langle T^0(\hat{u}|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)),v|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)\rangle=0. \tag{18}$$

Furthermore, $\forall v\in H^1(Y^{n-1})$,

$$\begin{aligned}\langle T^0u,v|_{\Gamma^0}\rangle&=\int_{Y^{n-1}}\nabla\hat{u}\cdot\nabla v+\sum_{\sigma\in\mathscr{A}_n}\langle T^\sigma\hat{u}|_{\Gamma^\sigma},v|_{\Gamma^\sigma}\rangle\\&=\int_{Y^{n-1}}\nabla\hat{u}\cdot\nabla v+\sum_{\sigma\in\mathscr{A}_n}\langle T^0(\hat{u}|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)),v|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)\rangle.\end{aligned} \tag{19}$$

We see that, for a fixed arbitrary integer $n>0$ (for example $n=1$), once the nonlocal operator T^0 is known, the restriction of $\mathscr{H}^0(u)$ to the truncated domain Y^{n-1} can be computed exactly by solving the boundary value problem (18) in Y^{n-1} with a boundary condition involving T^0. It is important to understand that (18) is a Poisson problem in Y^{n-1}, with Dirichlet boundary condition on Γ^0, homogeneous Neumann condition on $\partial Y^{n-1}\backslash(\Gamma^0\cup\Gamma^n)$, and for each $\sigma\in\mathscr{A}_n$,

$$\frac{\partial\hat{u}}{\partial n}+T^\sigma\hat{u}|_{\Gamma^\sigma}=0\quad\text{on }\Gamma^\sigma, \tag{20}$$

and all the operators T^σ, $\sigma\in\mathscr{A}_n$ are obtained readily from T^0 by (17). Eq. (20) is a nonlocal boundary condition, called *transparent boundary condition*. Transparent boundary conditions were proposed in computational physics for linear partial differential equations with constant coefficients. They allow the solution to be computed in a bounded domain without errors. They are particularly useful in electromagnetism, since one frequently deals with unbounded domains. There is a huge amount of literature on transparent boundary conditions, see [17] for one of the first papers.

Furthermore, solving (18) is equivalent to successively solving $1+2+\cdots+2^{n-1}$ boundary value problems in Y^0: indeed, an algorithm for solving (18) is as follows:

Algorithm 1

- Loop: for $p = 0$ to $n - 1$,
  - Loop: for $\sigma \in \mathscr{A}_p$ (at this point, $\hat{u}|_{\Gamma^\sigma}$ is known)
    - Solve the boundary value problem in Y^0: find $w \in H^1(Y^0)$ such that $w|_{\Gamma^0} = \hat{u}|_{\Gamma^\sigma} \circ \mathscr{M}_\sigma(F_1, F_2)$ and

$$\int_{Y^0} \nabla w \cdot \nabla v + \sum_{i=1}^{2} \langle T^0(w|_{F_i(\Gamma^0)} \circ F_i), v|_{F_i(\Gamma^0)} \circ F_i \rangle = 0, \quad \forall v \in \mathscr{V}(Y^0).$$

    - Set $\hat{u}|_{\mathscr{M}_\sigma(F_1,F_2)(Y^0)} = w \circ (\mathscr{M}_\sigma(F_1, F_2))^{-1}$.

Let us stress the fact that, in the numerical simulations of $\mathscr{H}^0(u)|_{Y^{n-1}}$, Algorithm 1 saves solving discrete boundary value problems in the domain Y^{n-1}, which is complicated when n is large. This is why the transparent boundary condition is well suited for numerical simulations, as soon as T^0 or a good approximation to T^0 is known. In Section 4.3, we review a method for approximating T^0 as the limit of an inductive sequence of operators, making use of (19).

4.3. An induction formula to approximate the Dirichlet–Neumann operator

Lemma 1, in the case $n = 1$, leads us to introduce the cone $\mathbb{O}$ of self-adjoint, positive semi-definite, bounded linear operators from $H^{\frac{1}{2}}(\Gamma^0)$ to its dual, vanishing on the constants, and the map $\mathbb{M} : \mathbb{O} \mapsto \mathbb{O}$ defined as follows: for $Z \in \mathbb{O}$, define $\mathbb{M}(Z)$ by $\forall u \in H^{\frac{1}{2}}(\Gamma^0),\ \forall v \in H^1(Y^0)$,

$$\langle \mathbb{M}(Z)u, v|_{\Gamma^0} \rangle = \int_{Y^0} \nabla \hat{u} \cdot \nabla v + \sum_{i=1}^{2} \langle Z(\hat{u}|_{F_i(\Gamma^0)} \circ F_i), v|_{F_i(\Gamma^0)} \circ F_i \rangle, \tag{21}$$

where $\hat{u} \in H^1(Y^0)$ is such that $\hat{u}|_{\Gamma^0} = u$ and

$$\forall v \in \mathscr{V}(Y^0), \quad \int_{Y^0} \nabla \hat{u} \cdot \nabla v + \sum_{i=1}^{2} \langle Z(\hat{u}|_{F_i(\Gamma^0)} \circ F_i), v|_{F_i(\Gamma^0)} \circ F_i \rangle = 0. \tag{22}$$

Remark 3. From the definition of $\mathbb{M}$, it can be seen that for all $p \geqslant 1$, if w satisfies the Poisson problem $\Delta w = 0$ in Y^{p-1}, with $\frac{\partial w}{\partial n} = 0$ on $\partial Y^{p-1} \backslash (\Gamma^0 \cup \Gamma^p)$, and with $\frac{\partial w}{\partial n}|_{\Gamma^\sigma} \circ \mathscr{M}_\sigma(F_1, F_2) = -2^p Z(w|_{\Gamma^\sigma} \circ \mathscr{M}_\sigma(F_1, F_2))$, $\sigma \in \mathscr{A}_p$, then $\frac{\partial w}{\partial n}|_{\Gamma^0} = \mathbb{M}^p(Z)(w|_{\Gamma^0})$.

Let us define the norm $\|\cdot\|$ on the space of linear and bounded operators from $H^{\frac{1}{2}}(\Gamma^0)$ to $(H^{\frac{1}{2}}(\Gamma^0))'$: $\|T\| = \sup_{v \in H^{\frac{1}{2}}(\Gamma^0), v \neq 0} \frac{\|Tv\|_{(H^{\frac{1}{2}}(\Gamma^0))'}}{\|v\|_{H^{\frac{1}{2}}(\Gamma^0)}}$. Lemma 1 tells that T^0 is a fixed point of $\mathbb{M}$. In fact, we have the following theorem, which says that T^0 can be approximated to an arbitrary accuracy by using an induction formula in $\mathbb{O}$:

Theorem 3. *The operator T^0 is the unique fixed point of $\mathbb{M}$. Moreover, for all $Z \in \mathbb{O}$, there exists a positive constant C such that, for all $r \geqslant 0$,*

$$\|\mathbb{M}^r(Z) - T^0\| \leqslant C\rho^{\frac{r}{4}},$$

where ρ, $0 < \rho < 1$ is the constant appearing in Theorem 2.

A typical loop for approximating T^0 to the accuracy ϵ is the following:

Algorithm 2
While $\|T^0 - Z\| > \epsilon$
$T^0 = Z$; $Z = \mathbb{M}(Z)$;

5. Propagation problems and transparent boundary conditions

The goal of this section is to study the weak solutions of the Helmholtz equation that are satisfied by a class of time-harmonic waves in the domain Ω^0. The analysis of the problem uses the compact imbedding of $H^1(\Omega^0)$ in $L^2(\Omega^0)$, see Theorem 1, and Fredholm's alternative. Next, as in Section 4.2, we are going to introduce transparent boundary conditions satisfied by the restriction of the solution to the truncated domain Y^0 (or Y^n). We shall make extensive use of self-similarity in order to design an approximation method for the operator entering this transparent condition. This method, which can be used for numerical simulations, is not as simple as the one for the Poisson problem, because the equation is not invariant by rescaling: as we go toward the finest structures of the ramified domain, diffusion effects dominate and the wave is exponentially damped. Besides, this is precisely why the approximation of the Dirichlet–Neumann operator will be possible.

5.1. The boundary value problem

For an integer $n \geqslant 0$, and for $\sigma \in \mathscr{A}_n$, given a real number k and $u \in H^{\frac{1}{2}}(\Gamma^\sigma)$ (with the notation $\Omega^\sigma = \mathscr{M}_\sigma(F_1,F_2)(\Omega^0)$, $\Gamma^\sigma = \mathscr{M}_\sigma(F_1,F_2)(\Gamma^0)$), let us consider the variational problem: find $\hat{u} \in H^1(\Omega^\sigma)$ such that

$$\hat{u}|_{\Gamma^\sigma} = u \text{ and } \forall v \in \mathscr{V}(\Omega^\sigma), \qquad \int_{\Omega^\sigma} \nabla\hat{u}\cdot\nabla v - k\int_{\Omega^\sigma}\hat{u}v = 0. \tag{23}$$

If it exists, $\hat{u}$ is a weak solution to the Helmholtz equation $\Delta\hat{u} + k\hat{u} = 0$ in Ω^σ.

Let us define the operator L_k^σ

$$L_k^\sigma : \mathscr{V}(\Omega^\sigma) \mapsto (\mathscr{V}(\Omega^\sigma))', \quad \langle L_k^\sigma w, v\rangle = \int_{\Omega^\sigma}\nabla w\cdot\nabla v - k\int_{\Omega^\sigma} wv. \tag{24}$$

A scaling argument yields that, for all $\sigma \in \mathscr{A}_n$, $v, w \in \mathscr{V}(\Omega^\sigma)$,

$$\langle L_k^\sigma w, v\rangle = \left\langle L^0_{\frac{k}{4^n}}(w\circ\mathscr{M}_\sigma(F_1,F_2)), v\circ\mathscr{M}_\sigma(F_1,F_2)\right\rangle. \tag{25}$$

Let us call $(\ker(L_k^\sigma))^\circ$ the closed subspace of $H^{\frac{1}{2}}(\Gamma^\sigma)$

$$(\ker(L_k^\sigma))^\circ = \left\{ u \in H^{\frac{1}{2}}(\Gamma^\sigma) \quad \text{s.t.} \left| \begin{array}{l} \forall \tilde{u} \in H^1(\Omega^\sigma) \quad \text{with } \tilde{u}|_{\Gamma^\sigma} = u,\ \forall v \in \ker(L_k^\sigma), \\ \int_{\Omega^\sigma}\nabla\tilde{u}\cdot\nabla v - k\tilde{u}v = 0 \end{array}\right. \right\}. \tag{26}$$

From the geometrical self-similarity, it can be verified that for all $\sigma \in \mathscr{A}_n$,

$$\left(\ker\left(L^0_{\frac{k}{4^n}}\right)\right)^\circ = \{u\circ\mathscr{M}_\sigma(F_1,F_2), u \in (\ker(L_k^\sigma))^\circ\}. \tag{27}$$

Proposition 2. *For all $n \in \mathbb{N}$, there exists a countable set $Sp^{D,n} = \{\lambda_p, p \in \mathbb{N}\}$ of positive numbers, with $\lambda_p \leqslant \lambda_{p+1}$ and $\lim_{p\to\infty}\lambda_p = +\infty$ such that for $\sigma \in \mathscr{A}_n$,*

- *for $k \in \mathbb{R}\setminus Sp^{D,n}$, the operator L_k^σ is one to one, with a bounded inverse,*
- *for all $k \in Sp^{D,n}$, $\ker(L_k^\sigma)$ has a positive and finite dimension.*

One can obtain a Hilbertian basis of $\mathscr{V}(\Omega^\sigma)$ by assembling bases of $\ker(L_k^\sigma)$, $k \in Sp^{D,n}$. We have

$$Sp^{D,n} = 4^n Sp^{D,0}. \tag{28}$$

For $u \in (\ker(L_k^\sigma))^\circ$ (see (26)), there exists $\hat{u} \in H^1(\Omega^\sigma)$ satisfying (23), and $\hat{u}$ is unique up to the addition of functions in $\ker(L_k^\sigma)$. Problem (23) defines an injective bounded operator $\mathscr{H}_k^\sigma$ from $(\ker(L_k^\sigma))^\circ$ to $H^1(\Omega^\sigma)/\ker(L_k^\sigma)$ by $\mathscr{H}_k^\sigma(u) = \hat{u}$.

Proof. Let us first focus on the case $n = 0$. The compact imbedding of $H^1(\Omega^0)$ in $L^2(\Omega^0)$, stated in Theorem 1, implies the existence of Sp^0 with the properties stated above. We have seen in Section 4 that $0 \notin Sp^{D,0}$. Then, for $n > 0$, identity (25) yields that $Sp^{D,n}$ given by (28) has the properties stated above. The last statement of Proposition 2 is a consequence of Fredholm's alternative. □

Remark 4. In relation with Proposition 2, we know from (28) that for any $k \in \mathbb{R}$, there exists a nonnegative integer $N(k) = \min\{n \in \mathbb{N} \text{ such that } \forall p \geqslant n, \ \forall \sigma \in \mathscr{A}_p, \ L_k^\sigma \text{ is coercive on } \mathscr{V}(\Omega^\sigma)\}$. We have $N(k) = 0$ if $k \leqslant 0$ and $N(k) \sim \log(k)$ as $k \to +\infty$.

From the geometrical self-similarity, it can be verified that for all $\sigma \in \mathscr{A}_n$,

$$\mathscr{H}_k^\sigma \circ \mathscr{M}_\sigma = \mathscr{H}^0_{\frac{k}{4^n}}. \tag{29}$$

Remark 5. One can prove the analogue of Lemma 2: there exist two positive constants k_0 and $\mu < 1$ such that, for all $k < k_0$, $\ker(L_k^0) = \{0\}$ and for all $u \in H^{\frac{1}{2}}(\Gamma^0)$, $\|\mathscr{H}_k^0(u)\|_{H^1(\Omega^1)} \leqslant \mu \|\mathscr{H}_k^0(u)\|_{H^1(\Omega^0)}$. From this, (29) and Remark 4, we have the analogue of Theorem 2: for all $u \in (\ker(L_k^0))^\circ$, $\|\mathscr{H}_k^0(u)\|_{H^1(\Omega^p)}$ decays exponentially with p as $p \to \infty$.

Remark 6. Similarly, for all $n \in \mathbb{N}$ and $\sigma \in \mathscr{A}_n$, the eigenvalues of the operator $\widetilde{L}^\sigma$

$$\widetilde{L}^\sigma : \ H^1(\Omega^\sigma) \mapsto (H^1(\Omega^\sigma))', \quad \langle \widetilde{L}^\sigma u, v\rangle = \int_{\Omega^\sigma} \nabla u \cdot \nabla v \tag{30}$$

form a nondecreasing sequence of nonnegative numbers $(\mu_p)_{p\in\mathbb{N}}$ with $\mu_0 = 0$, $\mu_1 > 0$, and $\lim_{p\to\infty}\mu_p = +\infty$. These numbers do not depend on σ. Calling $Sp^{N,n} = \{\mu_p, p \in \mathbb{N}\}$, we have $Sp^{N,n} = 4^n Sp^{N,0}$.

5.2. The Dirichlet–Neumann operators and transparent boundary conditions

For $\sigma \in \mathscr{A}_n$, $n \geqslant 0$, the Dirichlet–Neumann operator $T_k^\sigma : (\ker(L_k^\sigma))^\circ \mapsto ((\ker(L_k^\sigma))^\circ)'$ is defined by: $\forall u, v \in (\ker(L_k^\sigma))^\circ$,

$$\langle T_k^\sigma u, v\rangle = \int_{\Omega^\sigma} \nabla \mathscr{H}_k^\sigma(u) \cdot \nabla \tilde{v} - k \int_{\Omega^\sigma} \mathscr{H}_k^\sigma(u)\tilde{v} \tag{31}$$

for any function $\tilde{v} \in H^1(\Omega^\sigma)$ such that $\tilde{v}|_{\Gamma^\sigma} = v$. We have that, for all $u, v \in (\ker(L_k^\sigma))^\circ$,

$$\langle T_k^\sigma u, v\rangle = \left\langle T^0_{\frac{k}{4^n}}(u \circ \mathscr{M}_\sigma(F_1, F_2)), v \circ \mathscr{M}_\sigma(F_1, F_2)\right\rangle. \tag{32}$$

For $k \notin Sp^{D,n}$, T_k^σ is a bounded self-adjoint operator from $H^{\frac{1}{2}}(\Gamma^\sigma)$ to its dual.

For simplicity, since $\mathscr{A}_0$ has only one element, we use the notation $\mathscr{H}_k^0$ and T_k^0 if $n = 0$.

Lemma 2. *The operator T_k^σ is a perturbation of a bounded self-adjoint coercive operator from $(\ker(L_k^\sigma))^\circ$ to its dual by a compact operator.*

Lemma 3. *For all $u \in (\ker(L_k^0))^\circ$, the restriction $\hat{u}$ to Y^{n-1} of any function in the class $\mathscr{H}_k^0(u)$ satisfies, for all $\sigma \in \mathscr{A}_n$,*

$$\hat{u}|_{\Gamma^\sigma} \in (\ker(L_k^\sigma))^\circ \tag{33}$$

and

$$\hat{u}|_{\Gamma^0} = u, \quad \text{and} \quad \int_{Y^{n-1}} \nabla\hat{u} \cdot \nabla v - k \int_{Y^{n-1}} \hat{u} v + \sum_{\sigma\in\mathscr{A}_n} \langle T_k^\sigma \hat{u}|_{\Gamma^\sigma}, v|_{\Gamma^\sigma}\rangle = 0 \tag{34}$$

for all $v \in \mathscr{V}(Y^{n-1})$, such that for all $\sigma \in \mathscr{A}_n$, $v|_{\Gamma^\sigma} \in (\ker(L_k^\sigma))^\circ$. A solution to (33) and (34) can be extended to a solution to (23) in a unique manner. Problems (33) and (34) have a unique solution up to the addition of restrictions *of functions of* $\ker(L_k^0)$ *to* Y^{n-1}. *Furthermore,* $\forall v \in H^1(Y^{n-1})$, *such that* $v|_{\Gamma^0} \in (\ker(L_k^0))^\circ$, *and for all* $\sigma \in \mathscr{A}_n$, $v|_{\Gamma^\sigma} \in (\ker(L_k^\sigma))^\circ$,

$$\begin{aligned}\langle T_k^0 u, v|_{\Gamma^0}\rangle &= \int_{Y^{n-1}} \nabla\hat{u}\cdot\nabla v - k\int_{Y^{n-1}} \hat{u}v + \sum_{\sigma\in\mathscr{A}_n}\langle T_k^\sigma \hat{u}|_{\Gamma^\sigma}, v|_{\Gamma^\sigma}\rangle \\ &= \int_{Y^{n-1}} \nabla\hat{u}\cdot\nabla v - k\int_{Y^{n-1}} \hat{u}v + \sum_{\sigma\in\mathscr{A}_n}\Big\langle T^0_{\frac{k}{4^n}}(\hat{u}|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)), v|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)\Big\rangle. \end{aligned} \tag{35}$$

Proof. Since it greatly helps understanding what follows, in particular Algorithms 3 and 4 and Lemma 7, we choose to reproduce the proof contained in [1].

We skip the proof that the restriction of a function in $\mathscr{H}_k^0(u)$ to Y^{n-1} satisfies (33) and (34), since it is easily seen. We aim at proving that each solution to (33) and (34) can be extended in a unique manner to a solution of the original problem (23). This is clear if $\frac{k}{4^n} \notin Sp^{D,0}$. Thus, we consider the case when $\frac{k}{4^n} \in Sp^{D,0}$.

The subspace $\ker(L_k^\sigma)$ is finite dimensional. Furthermore, we can apply Holmgren's unique continuation theorem, see [30]: if $v \in \ker(L_k^\sigma)$ and if the normal derivative of v on Γ^σ is zero, then $v = 0$. Note that $\frac{\partial v}{\partial n}|_{\Gamma^\sigma} = 0$ if and only if the operator from $H^{\frac{1}{2}}(\Gamma^\sigma)$ to its dual: $w \mapsto \int_{\Omega^\sigma}\nabla\tilde{w}\cdot\nabla v - k\int_{\Omega^\sigma}\tilde{w}v$, where $\tilde{w}$ is any lifting of w in $H^1(\Omega^\sigma)$, is zero.

If the dimension of $\ker(L_k^\sigma)$ is $d > 0$, let $(\phi_{\sigma,i})_{i=1,\dots,d}$ be a basis of $\ker(L_k^\sigma)$. From the previous unique continuation result, we see that $\frac{\partial\phi_{\sigma,i}}{\partial n}|_{\Gamma^\sigma}$, $i = 1,\dots,d$ are linearly independent. This implies that $\left\{\left(\left\langle\frac{\partial\phi_{\sigma,i}}{\partial n}|_{\Gamma^\sigma},\psi\right\rangle\right)_{i=1,\dots,d}, \psi\in H^{\frac{1}{2}}(\Gamma^\sigma)\right\} = \mathbb{R}^d$. Therefore, there exists a family $(\psi_{\sigma,i})_{i=1,\dots,d}$ of linearly independent functions in $H^{\frac{1}{2}}(\Gamma^\sigma)$ such that $\int_{\Omega^\sigma}\nabla\tilde{\psi}_{\sigma,i}\cdot\nabla\phi_{\sigma,j} - k\int_{\Omega^\sigma}\tilde{\psi}_{\sigma,i}\phi_{\sigma,j} = \delta_{i,j}$, where $\tilde{\psi}_{\sigma,i}$ is an arbitrarily chosen function in $\mathscr{V}(\Omega^0)$ such that $\tilde{\psi}_{\sigma,i}|_{\Gamma^\sigma} = \psi_{\sigma,i}$ and $\tilde{\psi}_{\sigma,i}|_{\Omega^{\sigma'}} = 0$ for each $\sigma' \in \mathscr{A}_n$, $\sigma' \neq \sigma$. We have

$$\mathscr{V}(\Omega^0) = \{v\in\mathscr{V}(\Omega^0), v|_{\Gamma^\sigma}\in(\ker(L_k^\sigma))^\circ, \forall\sigma\in\mathscr{A}_n\} \oplus \left(\underset{\sigma\in\mathscr{A}_n}{\oplus}\operatorname{Span}(\tilde{\psi}_{\sigma,j}, j=1,\dots,d)\right). \tag{36}$$

Let $\hat{u}$ be a solution to (33) and (34): in order to extend it to a solution of (23), we have, for each $\sigma \in \mathscr{A}_n$, to choose the extension in $\mathscr{H}_k^\sigma(\hat{u}|_{\Gamma^\sigma})$, which is well defined since $\hat{u}|_{\Gamma^\sigma} \in (\ker(L_k^\sigma))^\circ$. With any such choice, the extended function belongs to $H^1(\Omega^0)$ and satisfies the Helmholtz equation in Y^{n-1} and Ω^σ, $\sigma \in \mathscr{A}_n$. But, for the extension to satisfy (23), its normal derivative must also be continuous across Γ^σ, $\sigma \in \mathscr{A}_n$. It can easily be seen that for each $\sigma \in \mathscr{A}_n$, there exists a unique function $\hat{u}_\sigma \in \mathscr{H}_k^\sigma(\hat{u}|_{\Gamma^\sigma})$, such that calling $\tilde{u}$ the extension of $\hat{u}$ by $\hat{u}_\sigma$ in Ω^σ, $\forall\sigma\in\mathscr{A}_n$, we have $\forall\sigma\in\mathscr{A}_n$, $\forall i = 1,\dots,d$, $\int_{\Omega^0}\nabla\tilde{u}\cdot\nabla\tilde{\psi}_{\sigma,i} - k\int_{\Omega^0}\tilde{u}\tilde{\psi}_{\sigma,i} = 0$. From this and from (33) and (34) and (36), we deduce that $\tilde{u}$ is a solution to (23).

The last two assertions of the lemma follow easily. □

We see from (32) and (34) that, if the nonlocal operator $T^0_{\frac{k}{4^n}}$ is known (which implies that $(\ker(L^0_{\frac{k}{4^n}})^\circ$ is known), then the restriction of $\mathscr{H}_k^0(u)$ to Y^{n-1}, $n \geqslant 1$, is characterized as the solution of a boundary value problem in Y^{n-1}, with a boundary condition involving $T^0_{\frac{k}{4^n}}$. Moreover, in the most frequent case when $\frac{k}{4^p} \notin Sp^{D,0}$, $\forall p = 0,\dots,n$, and if the operators $T^0_{\frac{k}{4^p}}$, $p = 1,\dots,n$ are known (or accurately approximated), then solving (34) is equivalent to solving successively $1 + 2 + \cdots + 2^{n-1}$ boundary value problems in Y^0 as follows:

Algorithm 3

- Loop: for $p = 0$ to $n - 1$,
  - Loop: for $\sigma \in \mathscr{A}_p$ (at this point, $\hat{u}|_{\Gamma^\sigma}$ is known)
    - Solve the boundary value problem in Y^0: find $w \in H^1(Y^0)$ such that $w|_{\Gamma^0} = \hat{u}|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)$ and $\forall v \in \mathscr{V}(Y^0)$,

$$\int_{Y^0}\left(\nabla w\cdot\nabla v - \frac{k}{4^p}wv\right) + \left\langle T^0_{\frac{k}{4^{p+1}}}(w|_{F_1(\Gamma^0)}\circ F_1), v|_{F_1(\Gamma^0)}\circ F_1\right\rangle + \left\langle T^0_{\frac{k}{4^{p+1}}}(w|_{F_2(\Gamma^0)}\circ F_2), v|_{F_2(\Gamma^0)}\circ F_2\right\rangle = 0.$$

    - Set $\hat{u}|_{\mathscr{M}_\sigma(F_1,F_2)(Y^0)} = w\circ(\mathscr{M}_\sigma(F_1,F_2))^{-1}$.

In the general case, if $\frac{k}{4^p} \in Sp^{D,0}$, for some p, $1 \leqslant p < n$, then, as in the proof of Lemma 3, additional finite dimensional linear systems must be solved at the step p of Algorithm 3, in order to enforce the continuity of the normal derivative of $\hat{u}$ at the interfaces Γ^σ, $\sigma \in \mathscr{A}_p$.

This method can be transposed at a discrete level (see Sections 6 and 7 for a related numerical method and simulations).

We are left with computing the operators $T^0_{\frac{k}{4^n}}$. Eqs. (32), (34) and (35) can be seen as a backward induction formula with respect to n, in order to compute T^0_k. The backward character of the induction makes the exact construction of T^0_k impossible. Yet, observing that $\lim_{n\to\infty} T^0_{\frac{k}{4^n}} = T^0$ (T^0 is the Dirichlet–Neumann operator for the Poisson problem, see Section 4.2) enables the initialization of the induction by approximating $T^0_{\frac{k}{4^n}}$ by T^0, for n large enough. The goal of what follows is to carry out this program in details.

5.3. Approximations of the Dirichlet–Neumann operators

5.3.1. An induction formula to approximate the Dirichlet–Neumann operators

For $\sigma \in \mathscr{A}_n$, $n \geqslant 0$ and $p \in \mathbb{N}$, $p \geqslant n$, let us introduce the operators $L^{\sigma,p}_k$

$$L^{\sigma,p}_k : \mathscr{V}(\Omega^\sigma) \mapsto (\mathscr{V}(\Omega^\sigma))', \qquad \langle L^{\sigma,p}_k u, v\rangle = \int_{\Omega^\sigma} \nabla u \cdot \nabla v - k \int_{Y^{p-1}\cap\Omega^\sigma} uv,$$

agreeing that $Y^{-1} = \emptyset$. Note that for $u \in H^{\frac{1}{2}}(\Gamma^\sigma)$, a function $\hat{u} \in H^1(\Omega^\sigma)$ such that

$$\hat{u}|_{\Gamma^\sigma} = u \quad \text{and} \quad \forall v \in \mathscr{V}(\Omega^\sigma), \qquad \int_{\Omega^\sigma} \nabla\hat{u} \cdot \nabla v - k \int_{Y^{p-1}\cap\Omega^\sigma} \hat{u} v = 0 \tag{37}$$

is a weak solution to the Helmholtz equation $\Delta\hat{u} + k 1_{Y^{p-1}\cap\Omega^\sigma}\hat{u} = 0$ in Ω^σ.

Let us call $(\ker(L^{\sigma,p}_k))^\circ$ the closed subspace of $H^{\frac{1}{2}}(\Gamma^\sigma)$:

$$(\ker(L^{\sigma,p}_k))^\circ = \left\{ u \in H^{\frac{1}{2}}(\Gamma^\sigma) \quad \text{s.t.} \left| \begin{array}{l} \forall \tilde{u} \in H^1(\Omega^\sigma) \quad \text{with } \tilde{u}|_{\Gamma^\sigma} = u, \ \ \forall v \in \ker(L^{\sigma,p}_k) \\ \int_{\Omega^\sigma} \nabla\tilde{u} \cdot \nabla v - k \int_{Y^{p-1}\cap\Omega^\sigma} \tilde{u} v = 0 \end{array} \right. \right\}. \tag{38}$$

We have the analogue of Proposition 2:

Proposition 3. *For all $n, p \in \mathbb{N}$ with $p \geqslant n$, there exists a countable set $Sp^{D,n,p} = \{\lambda_q, q \in \mathbb{N}\}$ of positive numbers, with $\lambda_q \leqslant \lambda_{q+1}$ and $\lim_{q\to\infty}\lambda_q = +\infty$ such that, for all $\sigma \in \mathscr{A}_n$,*

- *for all $k \in \mathbb{R} \setminus Sp^{D,n,p}$, the operator $L^{\sigma,p}_k$ is one to one, with a bounded inverse,*
- *for all $k \in Sp^{D,n,p}$, $\ker(L^{\sigma,p}_k)$ has a positive and finite dimension.*

We have

$$Sp^{D,n,p} = 4^n Sp^{D,0,p-n}. \tag{39}$$

If $u \in (\ker(L^{\sigma,p}_k))^\circ$, then there exists $\hat{u} \in H^1(\Omega^\sigma)$ satisfying (37), and $\hat{u}$ is unique up to functions in $\ker(L^{\sigma,p}_k)$. Problem (37) defines an injective bounded operator $\mathscr{H}^{\sigma,p}_k$ from $(\ker(L^{\sigma,p}_k))^\circ$ to $H^1(\Omega^\sigma)/\ker(L^{\sigma,p}_k)$ by $\mathscr{H}^\sigma_k(u) = \hat{u}$.

The modified Dirichlet–Neumann operator $T^{\sigma,p}_k : (\ker(L^{\sigma,p}_k))^\circ \mapsto ((\ker(L^{\sigma,p}_k))^\circ)'$ is defined by: $\forall u, v \in (\ker(L^{\sigma,p}_k))^\circ$,

$$\langle T^{\sigma,p}_k u, v\rangle = \int_{\Omega^\sigma} \nabla \mathscr{H}^{\sigma,p}_k(u) \cdot \nabla\tilde{v} - k \int_{Y^{p-1}\cap\Omega^\sigma} \mathscr{H}^{\sigma,p}_k(u)\tilde{v} \tag{40}$$

for any function $\tilde{v} \in H^1(\Omega^\sigma)$ such that $\tilde{v}|_{\Gamma^\sigma} = v$, and where $\mathscr{H}^{\sigma,p}_k(u)$ stands for any function in the class $\mathscr{H}^{\sigma,p}_k(u)$, if $k \in Sp^{D,n,p}$.

The analogues of Lemmas 2 and 3 are stated in the following lemma:

Lemma 4. *The operator $T^{\sigma,p}_k$ is the perturbation of a bounded and coercive self-adjoint operator from $(\ker(L^{\sigma,p}_k))^\circ$ to its dual by a compact operator.*

For all $n, p \in \mathbb{N}$, *with* $n \leqslant p$, $\sigma \in \mathscr{A}_n$, $u, v \in (\ker(L_k^{\sigma,p}))^\circ$, *we have* $u \circ \mathscr{M}_\sigma(F_1, F_2) \in \left(\ker\left(L_{\frac{k}{4^n}}^{0,p-n}\right)\right)^\circ$ *and*

$$\langle T_k^{\sigma,p} u, v\rangle = \left\langle T_{\frac{k}{4^n}}^{0,p-n}(u \circ \mathscr{M}_\sigma(F_1,F_2)), v \circ \mathscr{M}_\sigma(F_1,F_2)\right\rangle. \tag{41}$$

If $n \geqslant 1$, *then for all* $u \in (\ker(L_k^{0,p}))^\circ$, *the restriction* $\hat{u}$ *to* Y^{n-1} *of any function in the class* $\mathscr{H}_k^{0,p}(u)$ *satisfies the following boundary value problem: for all* $\sigma \in \mathscr{A}_n$,

$$\hat{u}|_{\Gamma^\sigma} \in (\ker(L_k^{\sigma,p}))^\circ, \tag{42}$$

and is a solution to the following boundary value problem: $\hat{u}|_{\Gamma^0} = u$, *and*

$$\hat{u}|_{\Gamma^0} = u, \quad \text{and} \quad \int_{Y^{n-1}} \nabla\hat{u}\cdot\nabla v - k\int_{Y^{n-1}} \hat{u}v + \sum_{\sigma\in\mathscr{A}_n} \langle T_k^{\sigma,p}(\hat{u}|_{\Gamma^\sigma}), v|_{\Gamma^\sigma}\rangle = 0. \tag{43}$$

$\forall v \in \mathscr{V}(Y^{n-1})$ *such that for all* $\sigma \in \mathscr{A}_n$, $v|_{\Gamma^\sigma} \in (\ker(L_k^{\sigma,p}))^\circ$, ((43) *can be written in terms of* $T_{\frac{k}{4^n}}^{0,p-n}$ *thanks to* (41)) *Problems* (42) *and* (43) *have a unique solution up to restrictions of functions of* $\ker(L_k^{0,p})$ *to* Y^{n-1}. *Furthermore,* $\forall v \in H^1(Y^{n-1})$, *such that* $v|_{\Gamma^0} \in (\ker(L_k^{0,r}))^\circ$ *and for all* $\sigma \in \mathscr{A}_n$, $v|_{\Gamma^\sigma} \in (\ker(L_k^{\sigma,p}))^\circ$,

$$\begin{aligned}\langle T_k^{0,p}u, v|_{\Gamma^0}\rangle &= \int_{Y^{n-1}} \nabla\hat{u}\cdot\nabla v - k\int_{Y^{n-1}} \hat{u}v + \sum_{\sigma\in\mathscr{A}_n}\langle T_k^{\sigma,p}\hat{u}|_{\Gamma^\sigma}, v|_{\Gamma^\sigma}\rangle \\ &= \int_{Y^{n-1}} \nabla\hat{u}\cdot\nabla v - k\int_{Y^{n-1}} \hat{u}v + \sum_{\sigma\in\mathscr{A}_n}\left\langle T_{\frac{k}{4^n}}^{0,p-n}(\hat{u}|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)), v|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)\right\rangle.\end{aligned} \tag{44}$$

If T^0 is available, then one can construct $T_k^{0,p}$ by the following induction:

Algorithm 4 (*The inductive construction of* $T_k^{0,p}$). Let us construct the closed subspaces $\mathscr{D}^{(j)}$ of $H^{\frac{1}{2}}(\Gamma^0)$ and the operators $(Z^{(j)})_{0\leqslant j\leqslant p}$ by

- $\mathscr{D}^{(0)} = H^{\frac{1}{2}}(\Gamma^0)$ and $Z^{(0)} = T^0$.
- *Induction formula* (*I.F.*). Suppose that after j steps, $j < p$, we have constructed the closed subspace $\mathscr{D}^{(j)}$ of $H^{\frac{1}{2}}(\Gamma^0)$ with finite codimension and the operator $Z^{(j)}$, from $\mathscr{D}^{(j)}$ to its dual, such that $Z^{(j)}$ is a perturbation of a coercive self-adjoint operator on $\mathscr{D}^{(j)}$ by a compact operator. We call $W^{(j)}$ the finite dimensional space containing the functions $w \in \mathscr{V}(Y^0)$, such that $w|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2) \in \mathscr{D}^{(j)}$, $\forall\sigma\in\mathscr{A}_1$, and

$$\int_{Y^0}\nabla w\cdot\nabla v - \frac{k}{4^{p-j-1}}\int_{Y^0} wv + \sum_{\sigma\in\mathscr{A}_1}\left\langle Z^{(j)}(w|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)), v|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)\right\rangle = 0,$$
$$\forall v \in \mathscr{V}(Y^0), \text{ with } v|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)\in\mathscr{D}^{(j)}, \forall\sigma\in\mathscr{A}_1.$$

We call $\mathscr{D}^{(j+1)}$ the closed subspace of $H^{\frac{1}{2}}(\Gamma^0)$ containing the functions v such that $\forall w \in W^{(j)}$,

$$\int_{Y^0}\nabla w\cdot\nabla \tilde{v} - \frac{k}{4^{p-j-1}}\int_{Y^0} w\tilde{v} + \sum_{\sigma\in\mathscr{A}_1}\left\langle Z^{(j)}(w|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)), \tilde{v}|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)\right\rangle = 0,$$
$$\forall \tilde{v} \in H^1(Y^0), \text{ with } \tilde{v}|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)\in\mathscr{D}^{(j)}, \forall\sigma\in\mathscr{A}_1, \text{ and } \tilde{v}|_{\Gamma^0} = v.$$

Then, from Fredholm's alternative, we know that the problem: find $\hat{u} \in H^1(Y^0)$ such that $\hat{u}|_{\Gamma^0} = u$, $\hat{u}|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2) \in \mathscr{D}^{(j)}$, $\forall\sigma\in\mathscr{A}_1$, and

$$\int_{Y^0}\nabla\hat{u}\cdot\nabla v - \frac{k}{4^{p-j-1}}\int_{Y^0}\hat{u}v + \sum_{\sigma\in\mathscr{A}_1}\left\langle Z^{(j)}(\hat{u}|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)), v|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)\right\rangle = 0$$
$$\forall v \in \mathscr{V}(Y^0) \text{ such that } \forall\sigma\in\mathscr{A}_1,\ v|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)\in\mathscr{D}^{(j)} \tag{45}$$

has a solution if $u \in \mathscr{D}^{(j+1)}$, which is unique up to functions in $W^{(j)}$. Then we can define the operator $Z^{(j+1)}$ from $\mathscr{D}^{(j+1)}$ to its dual, by: $\forall u, v \in \mathscr{D}^{(j+1)}$,

$$\left\langle Z^{(j+1)}u, v\right\rangle = \int_{Y^0}\nabla\hat{u}\cdot\nabla\tilde{v} - \frac{k}{4^{p-j-1}}\int_{Y^0}\hat{u}\tilde{v} + \sum_{\sigma\in\mathscr{A}_1}\left\langle Z^{(j)}(\hat{u}|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)), \tilde{v}|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)\right\rangle,$$
$$\forall \tilde{v} \in H^1(Y^0), \text{ with } \tilde{v}|_{\Gamma^\sigma}\circ\mathscr{M}_\sigma(F_1,F_2)\in\mathscr{D}^{(j)}, \forall\sigma\in\mathscr{A}_1, \text{ and } \tilde{v}|_{\Gamma^0} = v,$$

where $\hat{u}$ is a solution to (45). It can be seen that $Z^{(j+1)}$ has the same properties as $Z^{(j)}$.

Proposition 4. *The operators constructed by Algorithm 4 satisfy: for $j \leqslant p$,*

$$Z^{(j)} = T^{0,j}_{\frac{k}{4^{p-j}}} \quad \text{and} \quad \mathscr{D}^{(j)} = \left(\ker\left(L^{0,j}_{\frac{k}{4^{p-j}}}\right)\right)^{\circ}.$$

Remark 7. In fact, for k belonging to a dense subset in $\mathbb{R}$, the domains $\mathscr{D}^{(j)}$, $0 \leqslant j \leqslant p$ all coincide with $H^{\frac{1}{2}}(\Gamma^0)$.

Finally, the following result says that the original Dirichlet–Neumann operator T^σ_k can be approximated by the modified operator $T^{\sigma,p}_k$:

Theorem 4. *For $\sigma \in \mathscr{A}_n$ and $k \notin Sp^{D,n}$, there exists $P(k,n) \geqslant n$, such that for all $p \geqslant P(k,n)$, the operator $L^{\sigma,p}_k$ is one to one, and there exists a constant $C > 0$ (depending of k but not of n and p), such that, for $p \geqslant P(k,n)$,*

$$\|(L^{\sigma,p}_k)^{-1} - (L^\sigma_k)^{-1}\| \leqslant C 2^{-n-p}, \tag{46}$$

and the operator $T^{\sigma,p}_k$ is bounded from $H^{\frac{1}{2}}(\Gamma^\sigma)$ to its dual, with

$$\|T^{\sigma,p}_k - T^\sigma_k\| \leqslant C 2^{-n-p}. \tag{47}$$

Proof. Since $k \notin Sp^{D,n}$, L^σ_k is one to one. From (12), we have that $\|L^{\sigma,p}_k - L^\sigma_k\| \lesssim 2^{-n-p}$ and therefore $\lim_{p\to\infty} \|L^{\sigma,p}_k - L^\sigma_k\| = 0$. It is a standard matter to deduce the desired results from the previous observations. □

5.3.2. Stability of the approximation to the Dirichlet–Neumann operator

In practice, T^0 is not available, and one has to initialize the above mentioned backward induction by approximations of T^0. The approximation of T^0 is constructed by Algorithm 2. The following theorem gives an error estimate for the approximation of T^0_k:

Theorem 5. *For all $R \in \mathbb{O}$, $p, q \in \mathbb{N}$, consider the sequence $Z^{(n)}_{q,p}$, $0 \leqslant n \leqslant p$:*

- $Z^{(0)}_{q,p} = \mathbb{M}^q(R)$,
- *for $0 \leqslant j < p$, $Z^{(j+1)}_{q,p}$ is obtained from $Z^{(j))}_{q,p}$ by the induction (I.F.) in Algorithm 4,*

where $\mathbb{M}$ has been introduced in (21) and (22). Assume that $k \notin Sp^{D,0}$. Then there exist two integers $P(k)$ and $Q(k)$ such that for all $p > P(k)$, for all $q > Q(k) Z^{(p)}_{q,p}$ is a bounded operator from $H^{\frac{1}{2}}(\Gamma^0)$ to its dual, and there exists a constant C such that for all $p > P(k)$, $q > Q(k)$,

$$\|Z^{(p)}_{q,p} - T^0_k\| \leqslant C(\rho^{\frac{q}{4}} + 2^{-p}), \tag{48}$$

where $0 < \rho < 1$ is the constant introduced in (15).

6. A self-similar finite element method

6.1. Self-similar triangulations of Ω^0

To transpose the methods described above to finite element methods, one needs to use special triangulations of Ω^0. Before defining the triangulations, let us define, for an affine map G in $\mathbb{R}^2$ and a set of triangles $\mathscr{T}$, the set of triangles $G(\mathscr{T})$, obtained by transforming each triangle of $\mathscr{T}$ by the map G.

In what follows, we shall consider a regular family of triangulations $\mathscr{T}^0_h$ of Y^0 (see [6]), where the positive real number h stands for the maximal diameter of a triangle in $\mathscr{T}^0_h$, with the special property that the set of nodes of $\mathscr{T}^0_h$ lying on $F_1(\Gamma^0)$ (respectively $F_2(\Gamma^0)$) is the image by F_1 (resp. F_2) of the set of nodes of $\mathscr{T}^0_h$ lying on Γ^0.

Thanks to this special property, the sets of triangles

$$\mathscr{T}_h^n = \bigcup_{p=0}^{n} \bigcup_{\sigma \in \mathscr{A}_p} \mathscr{M}_\sigma(F_1, F_2)(\mathscr{T}_h^0)$$

form a regular family of triangulations of Y^n, because the intersection of two different triangles is either empty, or a common vertex to both triangles, or a common edge to both triangles, and because the regularity property is inherited from that of $\mathscr{T}_h^0$. Similarly, one can define triangulations of Ω^0 by

$$\mathscr{T}_h = \bigcup_{p=0}^{\infty} \bigcup_{\sigma \in \mathscr{A}_p} \mathscr{M}_\sigma(F_1, F_2)(\mathscr{T}_h^0).$$

The first two steps of the construction of $\mathscr{T}_h$ are depicted in Fig. 2.

6.2. Finite elements

Let us introduce the spaces of piecewise linear functions

$$V_h(Y^n) = \{v_h \in \mathscr{C}^0(\overline{Y^n}),\ \forall \tau \in \mathscr{T}_h^n,\ v_h|_\tau \text{ is linear}\} \subset H^1(Y^n), \qquad \mathscr{V}_h(Y^n) = V_h(Y^n) \cap \mathscr{V}(Y^n). \tag{49}$$

Similarly,

$$V_h(\Omega^0) = \{v_h \in H^1(\Omega^0),\ \forall \tau \in \mathscr{T}_h,\ v_h|_\tau \text{ is linear}\}, \qquad \mathscr{V}_h(\Omega^0) = V_h(\Omega^0) \cap \mathscr{V}(\Omega^0). \tag{50}$$

It is clear that for all $v_h \in V_h(\Omega^0)$, the restriction of v_h to Y^n belongs to $V_h(Y^n)$.

Let $V_h(\Gamma^n)$ (resp. $V_h(\Gamma^\sigma)$, $\sigma \in \mathscr{A}_n$) be the space of the traces of the functions of $V_h(\Omega^0)$ on Γ^n (resp. Γ^σ). It is clear that for all $v_h \in V_h(\Gamma^\sigma)$, $v_h \circ \mathscr{M}^\sigma(F_1, F_2) \in V_h(\Gamma^0)$.

We have the approximation result, whose proof is easy and skipped for brevity:

Lemma 5. *For all $v \in H^1(\Omega^0)$,*

$$\lim_{h\to 0} \inf_{v_h \in V_h(\Omega^0)} \|v - v_h\|_{H^1(\Omega^0)} = 0.$$

6.3. The finite element approximation to Poisson problems

6.3.1. Discrete Poisson problems and transparent boundary conditions

Given $u_h \in V_h(\Gamma^0)$, we consider the discrete counterpart of (13): find $w_h \in V_h(\Omega^0)$ such that

$$w_h|_{\Gamma^0} = u_h, \quad \int_{\Omega^0} \nabla w_h \cdot \nabla v_h = 0, \quad \forall v_h \in \mathscr{V}_h(\Omega^0). \tag{51}$$

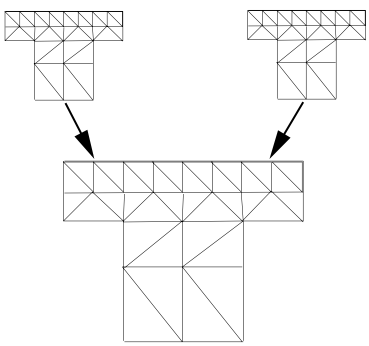


Fig. 2. The construction of the triangulation.

Exactly as in Proposition 1, problem (51) has a unique solution. This permits to define discrete harmonic lifting operator $\mathscr{H}_h^0 : V_h(\Gamma^0) \mapsto V_h(\Omega^0)$, $\mathscr{H}_h^0(u_h) = w_h$. A straightforward consequence of Céa's lemma (see [6]) is the following result:

Proposition 5. *Consider $u \in \mathscr{C}^0(\Gamma^0)$ such that $u \in V_h(\Gamma^0)$, for the whole family of meshes $\mathscr{T}_h$. We have*

$$\lim_{h\to 0} \|\mathscr{H}^0(u) - \mathscr{H}_h^0(u)\|_{H^1(\Omega^0)} = 0.$$

Remark 8. Proposition 5 can be much improved if global regularity results on $\mathscr{H}^0(u)$ are available. Obtaining such regularity results is an open problem to our knowledge. It will be the topic of forthcoming works.

We can also define the discrete Dirichlet–Neumann operator $T_h^0 : V_h(\Gamma^0) \mapsto (V_h(\Gamma^0))'$

$$\langle T_h^0 u_h, v_h\rangle = \int_{\Omega^0} \nabla\mathscr{H}_h^0(u_h)\cdot\nabla\mathscr{H}_h^0(v_h) = \int_{\Omega^0} \nabla\mathscr{H}^0(u_h)\cdot\nabla\tilde{v}_h \tag{52}$$

for any function $\tilde{v}_h \in V_h(\Omega^0)$ such that $\tilde{v}_h|_{\Gamma^0} = v_h$. If T_h^0 is available, one can apply the discrete analogue to Algorithm 1 in order to compute $\mathscr{H}_h^0(u_h)|_{Y^n}$, for any fixed integer $n \geqslant 0$. The next step is to compute T_h^0 or an accurate approximation of T_h^0. The construction is analogue to the one presented in Section 4.3. It is based on the following result, which can be proved in the same manner as for Theorem 2.

Theorem 6. *There exists a constant $\rho < 1$, independent of h such that for all $u_h \in V_h(\Gamma^0)$,*

$$\int_{\Omega^N} |\nabla\mathscr{H}_h^0(u_h)|^2 \leqslant \rho^N \int_{\Omega^0} |\nabla\mathscr{H}_h^0(u_h)|^2. \tag{53}$$

Exactly as for the continuous problem, we introduce the cone $\mathbb{O}_h$ of self adjoint, positive semi-definite, bounded linear operators from $V_h(\Gamma^0)$ to its dual, vanishing on the constants, and the map $\mathbb{M}_h : \mathbb{O}_h \mapsto \mathbb{O}_h$ defined as follows: for $Z_h \in \mathbb{O}_h$, define $\mathbb{M}_h(Z_h)$ by $\forall u_h \in V_h(\Gamma^0)$, $\forall v_h \in V_h(Y^0)$,

$$\langle \mathbb{M}_h(Z_h)u_h, v_h|_{\Gamma^0}\rangle = \int_{Y^0} \nabla\hat{u}_h\cdot\nabla v_h + \sum_{i=1}^{2}\langle Z_h(\hat{u}_h|_{F_i(\Gamma^0)}\circ F_i), v_h|_{F_i(\Gamma^0)}\circ F_i\rangle, \tag{54}$$

where $\hat{u}_h \in V_h(Y^0)$ is such that $\hat{u}_h|_{\Gamma^0} = u_h$ and

$$\forall v_h \in V_h(Y_0) \quad \text{with } v_h|_{\Gamma_0} = 0, \quad \int_{Y^0} \nabla\hat{u}_h\cdot\nabla v_h + \sum_{i=1}^{2}\langle Z_h(\hat{u}_h|_{F_i(\Gamma^0)}\circ F_i), v_h|_{F_i(\Gamma^0)}\circ F_i\rangle = 0. \tag{55}$$

The analogue of Theorem 3 is:

Theorem 7. *The operator T_h^0 is the unique fixed point of $\mathbb{M}_h$ and for all $Z_h \in \mathbb{O}_h$, there exists a positive constant C independent of n such that, for all $n \geqslant 0$,*

$$\|\mathbb{M}_h^n(Z_h) - T_h^0\| \leqslant C\rho^{\frac{n}{4}}, \tag{56}$$

where ρ, $0 < \rho < 1$ is the constant appearing in Theorem 6.

Theorem 7 says that the discrete counterpart of Algorithm 2 can be used for computing T_h^0.

6.3.2. The numerical implementation of the induction formula

The central part of Algorithm 2 is the computation of $\mathbb{M}_h(Z_h)$ for $Z_h \in \mathbb{O}_h$.

Let us call $N_h(Y^0)$ (resp. N) the dimension of $V_h(Y^0)$ (resp. $V_h(\Gamma^0)$). Call $(x_i)_{i=1,\dots,N}$ the abscissa of the mesh-nodes lying on Γ^0, ordered increasingly. Let us introduce the nodal basis $(\phi_i)_{i=1,\dots,N_h(Y^0)}$ of $V_h(Y^0)$ ordered as follows:

1. for $j = 1,\dots,N$, ϕ_j corresponds to the node $(x_j, 0) \in \Gamma^0$.
2. for $i = 1,2$ and $j = 1,\dots,N$, ϕ_{iN+j} corresponds to the node $F_i(x_j, 0) \in F_i(\Gamma^0)$.
3. for $3N < j \leqslant N_h(Y^0)$, the node corresponding to ϕ_j belongs to $\overline{Y^0} \setminus (\Gamma^0 \cup \Gamma^1)$.

Consider the bilinear for $a_h : V_h(Y^0) \times V_h(Y^0) \mapsto \mathbb{R}$: $a_h(u_h, v_h) = \int_{Y^0} \nabla u_h \cdot \nabla v_h$, and let A be the matrix of a_h in the nodal basis described above. We have the block decomposition

$$A = \begin{pmatrix} A_{\Gamma^0,\Gamma^0} & 0 & A_{\Gamma^0,I} \\ 0 & A_{\Gamma^1,\Gamma^1} & A_{\Gamma^1,I} \\ A^{\mathrm{T}}_{\Gamma^0,I} & A^{\mathrm{T}}_{\Gamma^1,I} & A_{I,I} \end{pmatrix}, \quad \begin{matrix} A_{\Gamma^0,\Gamma^0} \in \mathbb{R}^{N\times N}, \\ A_{\Gamma^1,\Gamma^1} \in \mathbb{R}^{2N\times 2N}. \end{matrix} \tag{57}$$

The block $A_{I,I}$ is positive definite; it is the matrix arising when dealing with a Poisson problem with Dirichlet conditions on $\Gamma^0 \cup \Gamma^1$ and Neumann conditions on $\partial Y^0 \backslash (\Gamma^0 \cup \Gamma^1)$. The Schur complement of A obtained by eliminating the degrees of freedom corresponding to the mesh nodes in $\overline{Y^0} \setminus (\Gamma^0 \cup \Gamma^1)$ is $S \in R^{3N\times 3N}$

$$S = \begin{pmatrix} S_{\Gamma^0,\Gamma^0} & S_{\Gamma^0,\Gamma^1} \\ S^{\mathrm{T}}_{\Gamma^0,\Gamma^1} & S_{\Gamma^1,\Gamma^1} \end{pmatrix}, \quad \begin{matrix} S_{\Gamma^0,\Gamma^0} = A_{\Gamma^0,\Gamma^0} - A_{\Gamma^0,I}A^{-1}_{I,I}A^{\mathrm{T}}_{\Gamma^0,I} \in \mathbb{R}^{N\times N}, \\ S_{\Gamma^1,\Gamma^1} = A_{\Gamma^1,\Gamma^1} - A_{\Gamma^1,I}A^{-1}_{I,I}A^{\mathrm{T}}_{\Gamma^1,I} \in \mathbb{R}^{2N\times 2N}, \\ S_{\Gamma^0,\Gamma^1} = -A_{\Gamma^0,I}A^{-1}_{I,I}A^{\mathrm{T}}_{\Gamma^1,I} \in \mathbb{R}^{N\times 2N}. \end{matrix} \tag{58}$$

Denoting O the cone of the positive semi-definite matrices $Z \in \mathbb{R}^{N\times N}$ such that for $i = 1, \ldots, N$, $\sum_{j=1}^{N} Z_{ij} = 0$, it is clear from the interpretations of S_{Γ^0,Γ^0}, S_{Γ^1,Γ^1} and S_{Γ^0,Γ^1} given above that the matrix counterpart of the operator $\mathbb{M}_h$ defined in (54) and (55) is the operator $M : O \mapsto O$:

$$M(Z) = S_{\Gamma^0,\Gamma^0} - S_{\Gamma^0,\Gamma^1}\left(S_{\Gamma^1,\Gamma^1} + \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix}\right)^{-1} S^{\mathrm{T}}_{\Gamma^0,\Gamma^1}. \tag{59}$$

As a corollary to Theorem 7, we have the

Proposition 6. *For any $Z \in O$, the sequence $M^n(Z)$ converges geometrically to the unique fixed point T of M, and T is the matrix of the discrete Dirichlet–Neumann operator T^0_h defined in (52) in the nodal basis of $V_h(\Gamma^0)$.*

Proposition 6 tells us that, for obtaining an approximation of the matrix T with an accuracy ϵ (in a fixed matrix norm), one can start from any matrix $Z \in O$, ($Z = 0$ is possible) and repeat $M(Z) \leftarrow Z$, $\mathrm{O}(|\log \epsilon|)$ times. Assuming that S_{Γ^0,Γ^0}, S_{Γ^1,Γ^1} and S_{Γ^0,Γ^1} are known, and performing a Cholesky factorization of

$$S_{\Gamma^1,\Gamma^1} + \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix},$$

the map $M(Z) \leftarrow Z$ requires $\mathrm{O}(N^3)$ operations.

Thanks to the sparsity of the matrix A, and using an efficient algorithm for factorizing A_{II} (for example SuperLU), one may compute the matrices S_{Γ^0,Γ^0}, S_{Γ^1,Γ^1} and S_{Γ^0,Γ^1} in $\mathrm{O}(N^3)$ operations. This has to be done once and for all. Finally, one may approach T to an accuracy ϵ with a work of $\mathrm{O}(|\log \epsilon|)N^3$ operations.

6.4. The finite element approximation to Helmholtz problems

We are interested in the discrete version of (23): find $\hat{u} \in V_h(\Omega^0)$ such that

$$\hat{u}_h|_{\Gamma^\sigma} = u_h \quad \text{and} \quad \forall v \in \mathscr{V}_h(\Omega^\sigma), \quad \int_{\Omega^\sigma} \nabla \hat{u}_h \cdot \nabla v_h - k \int_{\Omega^0} \hat{u}_h v = 0. \tag{60}$$

All the results proved in Section 5 have their discrete counterparts, as soon as the discrete analogous of Holmgren's theorem (used in the proofs of Lemmas 3 and 4) is true (such a result may not hold for any mesh and wavenumber, but in practice, it is almost ever true).

Here, we do not discuss the convergence of the discrete method when the step h tends to 0. For that, one has to prove the convergence of the spectrum of the discrete problem to the spectrum of the continuous problem ($Sp^{D,n}$ introduced in Section 5.1). This may be done by using the results contained in [6]. Instead, let us discuss the analogue of the induction formula (I.F.) in Algorithm 4, working directly at the matrix level with the same notations as in Section 6.3.2. The matrix of the bilinear form $V_h(Y^0) \times V_h(Y^0) \mapsto \mathbb{R}$: $(u_h, v_h) \mapsto \int_{Y^0} \nabla u_h \cdot \nabla v_h - k \int_{Y^0} u_h v_h$ in the nodal basis is $A - kB$ where A is the stiffness matrix introduced in Section 6.3.2 and where B is the mass matrix. Both A and B have the block decomposition described in (57).

The counterpart of problem (45) is: given $U \in \mathbb{R}^N$, to find $\widehat{U}_I$ and $\hat{U}_{\Gamma^1}$ such that

$$\left(\begin{pmatrix} A_{\Gamma^1,\Gamma^1} & A_{\Gamma^1,I} \\ A^{\mathrm{T}}_{\Gamma^1,I} & A_{I,I} \end{pmatrix} - k4^{j-p+1}\begin{pmatrix} B_{\Gamma^1,\Gamma^1} & B_{\Gamma^1,I} \\ B^{\mathrm{T}}_{\Gamma^1,I} & B_{I,I} \end{pmatrix} + \begin{pmatrix} \widetilde{Z}^{(j)} & 0 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} \hat{U}_{\Gamma^1} \\ \hat{U}_I \end{pmatrix} = -\begin{pmatrix} 0 \\ (A^{\mathrm{T}}_{\Gamma^0,I} - k4^{j-p+1}B^{\mathrm{T}}_{\Gamma^0,I})U \end{pmatrix}, \tag{61}$$

where

$$\widetilde{Z}^{(j)} = \begin{pmatrix} Z^{(j)} & 0 \\ 0 & Z^{(j)} \end{pmatrix}.$$

Let us assume that the real number k is such that, for all j, $0 \leqslant j \leqslant p$, the matrix

$$G^{(j)}_k = \begin{pmatrix} A_{\Gamma^1,\Gamma^1} - k4^{j-p+1}B_{\Gamma^1,\Gamma^1} + \widetilde{Z}^{(j)} & A_{\Gamma^1,I} - k4^{j-p+1}B_{\Gamma^1,I} \\ A^{\mathrm{T}}_{\Gamma^1,I} - k4^{j-p+1}B^{\mathrm{T}}_{\Gamma^1,I} & A_{I,I} - k4^{j-p+1}B_{I,I} \end{pmatrix}$$

in the left hand side of (61) is invertible. This occurs for k in a dense subset of $\mathbb{R}$. Then the discrete counterpart of the induction formula (I.F.) is

$$Z^{(j+1)} = A_{\Gamma^1,\Gamma^1} - \frac{k}{4^{p-j-1}}B_{\Gamma^1,\Gamma^1} - \left(0, A_{\Gamma^0,I} - \frac{k}{4^{p-j-1}}B_{\Gamma^0,I}\right)\left(G^{(j)}_k\right)^{-1}\begin{pmatrix} 0 \\ A^{\mathrm{T}}_{\Gamma^0,I} - \frac{k}{4^{p-j-1}}B^{\mathrm{T}}_{\Gamma^0,I} \end{pmatrix}. \tag{62}$$

We use the following algorithm in order to approximate the discrete version of T^0_k: for all $R \in O$, $p, q \in \mathbb{N}$, we consider the sequence $Z^{(j)}_{q,p}$, $0 \leqslant j \leqslant p$:

- $Z^{(0)}_{q,p} = M^q(R)$ where M has been introduced in (59),
- for $0 \leqslant j < p$, $Z^{(j+1)}_{q,p}$ is obtained from $Z^{(j))}_{q,p}$ by the induction (62).

7. Numerical results

A software has been written in C++ for implementing the methods described above.

In the numerical tests, we have taken for Ω^0 a dilation by the factor π of the domain described in Section 2. The mesh used for Y^0 is plotted in Fig. 3. It has the property described in Section 6.1, which permits the construction of a self-similar mesh of Ω^0. Note that there are 40 mesh nodes on Γ^0, i.e. $N = 40$.

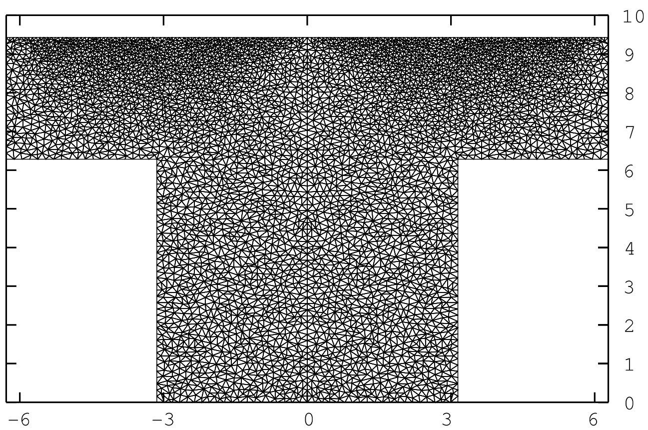


Fig. 3. The mesh used for Y^0.

7.1. Computation of T^0

The first thing to do is to compute T^0 ((more exactly its discrete counterpart) by applying Algorithm 2: we compute it for both the Laplace operator and the operator div$\chi\nabla$ where

$$\chi = \begin{pmatrix} 1 & 0 \\ 0 & 100 \end{pmatrix}.$$

Note that since the number of nodes on Γ^0 is 40, the discrete Dirichlet–Neumann operator can be represented by a dense square matrix of order 40.

In Fig. 4, we plot the Frobenius norm of $M^{q+1}(0) - M^q(0)$ for the two cases above. We see that the norm of the increment $M^{q+1}(0) - M^q(0)$ decays exponentially as $q \to \infty$ and that the decay factor is quite small (of the order $10\times^{-6}$ in the case of the Laplace operator and 10^{-4} in the anisotropic case). We see that a few iterations are enough to have a very accurate approximation of the discrete version of T^0.

7.2. Computation of T_k^0 for $k = 1$

We take $k = 1$, and we approximate the discrete version of T_k^0 by the construction given at the end of Section 6.4. We choose $q = 4$ because the numerical tests in Section 7.1 show that four iterations of the fixed point Algorithm 2 are enough for computing T_h^0. Then we test the method for $p \leqslant 27$. There is no point in taking larger values of p when working in double precision, because 4^{-27} is of the order of the machine smallest

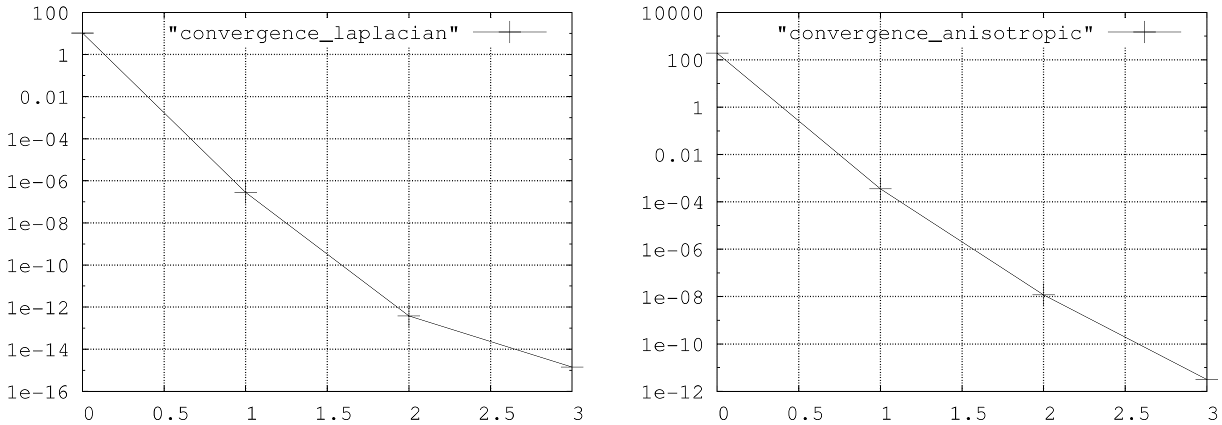


Fig. 4. The Frobenius norms of the increments $M^{q+1}(0) - M^q(0)$ (in log-scale) vs. q.

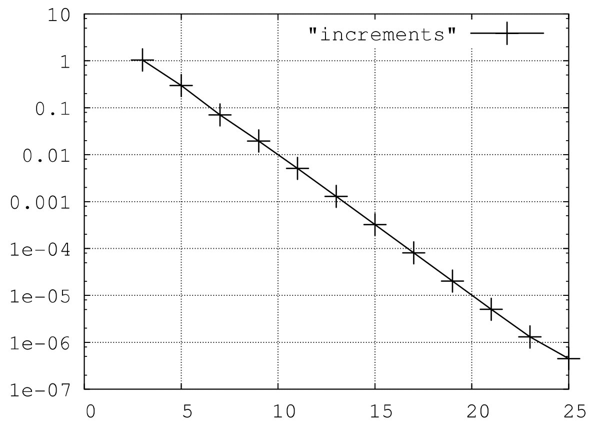


Fig. 5. The Frobenius norms of the increments $Z_{4,p}^{(p)} - Z_{4,p-2}^{(p-2)}$ (in log-scale) as a function of p.

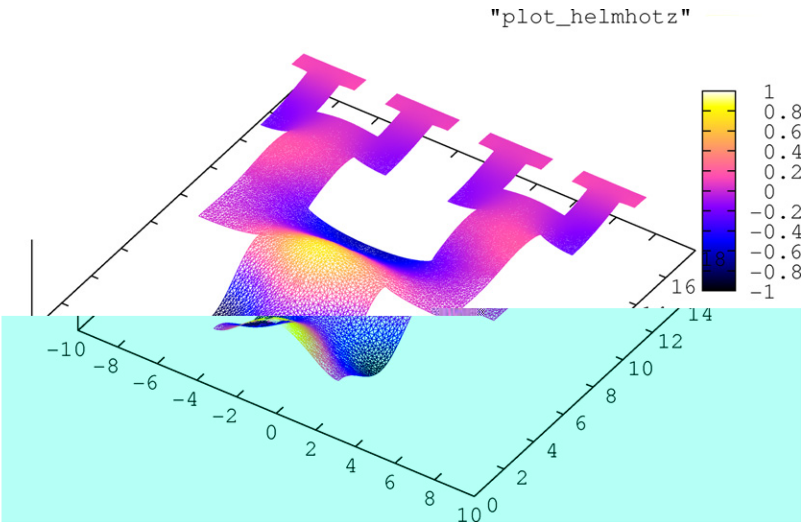


Fig. 6. $(\mathscr{H}^0_k u)|_{Y^2}$ with $k = 1$ and $u = \frac{(x_1^2-\pi^2)^2}{\pi^4}$.

double precision number. In Fig. 5 we plot the Frobenius norm of the increments $Z^{(p)}_{4,p} - Z^{(p-2)}_{4,p-2}$ as a function of p. We see that these increments decay exponentially in n, and the decay exponent is very close to $\frac{1}{2}$ (in log-scale, the graph is very close to a straight line, with a slope close to $-\log(2)$). Fig. 5 shows that for approximating T^0_k with an error of order 10^{-6}, we need approximately 25 iterations of the construction above.

7.3. The Helmholtz equation

We have used Algorithm 3, and the approximations of $T^0_{\frac{k}{4^p}}$, $p = 1, 2, 3$ computed with the discrete analogue of Algorithm 4, in order to compute numerically $\mathscr{H}^0_k u$ where $k = 1$ and $u = \frac{(x_1^2-\pi^2)^2}{\pi^4}$, in Y^2: the result is plotted in Fig. 6. Again, we stress the fact that for obtaining the result, we did not solve a boundary value problem in Y^2, but rather, seven boundary problems in Y^0. Nevertheless, the function matches smoothly at the interfaces Γ^σ, $\sigma \in \mathscr{A}_1 \cup \mathscr{A}_2$.

8. Application: the vibration modes and numerical simulations of the wave equation

The goal here is the computation of the eigenvalues and normalized eigenmodes of the Neumann operator $\widetilde{L}^0$ introduced in (30). For what follows, we define the operator $\widetilde{L}^0_k : H^1(\Omega^0) \mapsto (H^1(\Omega^0))'$ by $\langle \widetilde{L}^0_k w, v\rangle = \int_{\Omega^0} \nabla w \cdot \nabla v - k \int_{\Omega^0} wv$, for all v, w in $H^1(\Omega^0)$. This operator is naturally associated with the Helmholtz equation $\Delta u + ku = 0$ in Ω^0 and Neumann boundary condition on $\partial\Omega^0$.

8.1. Characterization of the eigenvalues of the Neumann problem

The following lemmas will be useful for computing the eigenvalues in $Sp^{N,0}$.

Lemma 6. *For any real number k, if $u \in \ker(\widetilde{L}^0_k)$, then $u|_{\Gamma^0} \in (\ker(L^0_k))^\circ$.*

Proof. We have $\int_{\Omega^0} \nabla u \cdot \nabla v - kuv = 0$ for all $v \in H^1(\Omega^0)$. Let $\tilde{u} \in H^1(\Omega^0)$ be another lifting of $u|_{\Gamma^0}$, then $e = u - \tilde{u} \in \mathscr{V}(\Omega^0)$ and for all $v \in \ker(L^0_k)$, $\int_{\Omega^0} \nabla e \cdot \nabla v - kev = 0$. Subtracting the last two integral identities, we obtain that for all $v \in \ker(L^0_k)$, $\int_{\Omega^0} \nabla\tilde{u} \cdot \nabla v - k\tilde{u}v = 0$. This exactly says that $u|_{\Gamma^0} \in (\ker(L^0_k))^\circ$. □

Lemma 7

$$Sp^{N,0} = \{k \in \mathbb{R},\ \textit{such that}\ \ker(T^0_k) \neq \{0\}\}, \tag{63}$$

and

$$\forall u \in \ker(T_k^0),\ \textit{there exists a unique}\ \hat{u} \in \ker(\widetilde{L}_k^0) \cap \mathscr{H}_k^0(u). \tag{64}$$

One can obtain a Hibertian basis of $H^1(\Omega^0)$ *by assembling bases of* $\mathscr{H}_k^0(\ker(T_k^0)) \cap \ker(\widetilde{L}_k^0)$ *for* $k \in Sp^{N,0}$.

Proof. We know that $k \in Sp^{N,0}$ if and only if there exists $\hat{u} \in H^1(\Omega^0)$, $\hat{u} \neq 0$, such that

$$\int_{\Omega^0} \nabla\hat{u} \cdot \nabla v = k \int_{\Omega^0} \hat{u}v, \quad \forall v \in H^1(\Omega^0). \tag{65}$$

Calling $u \in H^{\frac{1}{2}}(\Gamma^0)$ the trace of $\hat{u}$ on Γ^0, Lemma 6 tells us that $u \in (\ker(L_k^0))^\circ$. So $T_k^0 u$ is well defined. Holmgren's theorem tells us that $u \neq 0$. Furthermore, remarking that $\hat{u}$ is a function in the class $\mathscr{H}_k^0(u)$, we see from the definition (31) of T_k^0 that $T_k^0 u = 0$.

Conversely, if $u \in \ker(T_k^0)$, it can be proved as in the proof of Lemma 3 that there exists a unique function $\hat{u}$ in the class $\mathscr{H}_k^0(u)$ which satisfies (65). We have proved (63) and (64). The last statement of the lemma follows. □

From Lemma 7, one can compute the eigenmodes of $\widetilde{L}^0$ by searching the wavenumbers k such that T_k^0 is noninjective, and by taking the harmonic lifting $\mathscr{H}_k^0$ of the vectors belonging to the kernel of T_k^0 (in the rare case when $k \in Sp^{D,0} \cap Sp^{N,0}$, if $u \in \ker(T_k^0)$, then $\mathscr{H}_k^0(u)$ is a class of functions given up to the addition of functions in $\ker(L_k^0)$, so, in order to obtain the harmonic lifting of u in $\ker(\widetilde{L}^0)$, one needs to solve an auxiliary linear system in $\mathbb{R}^d$, where d is the dimension of $\ker(L_k^0)$ – note that we have never experienced this case in our computations). Of course, if the discrete version of Holmgren's theorem holds, it is possible to carry out this program with the self-similar finite element discretization introduced above, because Lemmas 6 and 7 have their discrete counterparts.

Remark 9. Conversely, it is possible to compute the eigenmodes of the Dirichlet operator L^0 by studying the Neumann–Dirichlet operators related to the Helmholtz equation.

8.2. Normalization of the eigenmodes of the Neumann problem

A more difficult task is to obtain eigenmodes with unit $L^2(\Omega^0)$-norm, in order to construct an orthonormal basis of $L^2(\Omega^0)$. For example, as we shall see in Section 7, this is important for projecting a compactly supported function on eigenspaces. One must recall that it is not possible to compute the eigenmodes in the whole domain Ω^0; what is possible, by using the method presented above, is to compute their restriction to Y^n, where n is a nonnegative integer. Thus, trying to obtain the $L^2(\Omega^0)$ norm of an eigenmode seems impossible. Yet, the following result says that, when $k \notin Sp^{D,0}$, it is possible to normalize the eigenmodes by a perturbation method.

Proposition 7. *Consider* $k \in Sp^{N,0}$ *such that* $k > 0$ *and* $k \notin Sp^{D,0}$, $u \in \ker(T_k^0)$, $u \neq 0$. *Let* δk *be a small variation of* k, *such that* $k + \delta k \notin Sp^{D,0}$:

$$\delta k \|\mathscr{H}_k^0(u)\|^2_{L^2(\Omega^0)} = -\langle T^0_{k+\delta k}u, u\rangle + o(\delta k). \tag{66}$$

For δk *small enough,* $\delta k\langle T^0_{k+\delta k}u, u\rangle < 0$ *and*

$$\|\mathscr{H}_k^0(u)\|^2_{L^2(\Omega^0)} = -\lim_{\delta k \to 0} \frac{\langle T^0_{k+\delta k}u, u\rangle}{\delta k}. \tag{67}$$

Proof. For simplicity, we note $\hat{u} = \mathscr{H}_k^0(u)$. Let $\hat{e} \in \mathscr{V}(\Omega^0)$ be such that $\hat{u} + \hat{e} = \mathscr{H}^0_{k+\delta k}(u)$. Since $u \in \ker(T_k^0)$ and $\hat{u} = \mathscr{H}_k^0(u)$, we know that for all $v \in H^1(\Omega^0)$, $\int_{\Omega^0} \nabla\hat{u} \cdot \nabla v - k\hat{u}v = 0$. In particular, for $v = \hat{u} + \hat{e}$,

$$\int_{\Omega^0} \nabla\hat{u} \cdot \nabla(\hat{u} + \hat{e}) - k \int_{\Omega^0} \hat{u}(\hat{u} + \hat{e}) = 0. \tag{68}$$

On the other hand, we know that for all $v \in \mathscr{V}(\Omega^0)$, $\int_{\Omega^0} \nabla(\hat{u}+\hat{e}) \cdot \nabla v - (k+\delta k)\int_{\Omega^0}(\hat{u}+\hat{e})v = 0$. In particular, for $v = \hat{e}$,

$$\int_{\Omega^0} \nabla\hat{e} \cdot \nabla(\hat{u}+\hat{e}) - (k+\delta k)\int_{\Omega^0} \hat{e}(\hat{u}+\hat{e}) = 0. \tag{69}$$

Adding (68) and (69) yields $\int_{\Omega^0} |\nabla(\hat{u}+\hat{e})|^2 - (k+\delta k)\int_{\Omega^0}|\hat{u}+\hat{e}|^2 = -\delta k \int_{\Omega^0} \hat{u}(\hat{u}+\hat{e})$. But, since $\hat{u}+\hat{e} = \mathscr{H}^0_{k+\delta k}(u)$,

$$\int_{\Omega^0} |\nabla(\hat{u}+\hat{e})|^2 - (k+\delta k)\int_{\Omega^0} |\hat{u}+\hat{e}|^2 = \langle T^0_{k+\delta k}(\hat{u}+\hat{e})|_{\Gamma^0}, (\hat{u}+\hat{e})|_{\Gamma^0}\rangle = \langle T^0_{k+\delta k}u, u\rangle.$$

Therefore, $\delta k\|\hat{u}\|^2_{L^2(\Omega^0)} = -\langle T^0_{k+\delta k}u, u\rangle - \delta k\int_{\Omega^0}\hat{u}\hat{e}$. Finally, since $k \notin Sp^{D,0}$, we deduce from the equation $\int_{\Omega^0}\nabla\hat{e}\cdot\nabla v - k\int_{\Omega^0}\hat{e}v = \delta k\int_{\Omega^0}(\hat{u}+\hat{e})v$, $\forall v \in \mathscr{V}(\Omega^0)$, that $\|\hat{e}\|_{L^2(\Omega^0)} = O(\delta k)$, which completes the proof. □

Proposition 7 permits the scaling of the vibration modes obtained by the characterization in Section 8.1 so that their $L^2(\Omega^0)$ norm is close to one (not exactly one, since the scaling factor is obtained by a perturbation method). This will allow a function to be projected accurately enough on the eigenspaces.

Remark 10. We do not know how to normalize the vibration mode when $k \in Sp^{N,0} \cap Sp^{D,0}$. However, we have not observed this situation in our computations.

8.3. The projection of a compactly supported function on the space spanned by the first P eigenmodes

Let $(e_p)_{p=0,\ldots,\infty}$ be a Hilbertian basis of $L^2(\Omega^0)$ made of eigenmodes of $\widetilde{L}^0$ with unit $L^2(\Omega^0)$ norm. In the following, we call k_p the eigenvalue of $\widetilde{L}^0$ corresponding to e_p. We also call Λ^P the subspace of $H^1(\Omega^0)$: $\Lambda^P = \text{span}(e_p)_{p=0,\ldots,P}$.

Assume that with the method described in Section 8.2, we have obtained eigenmodes $\tilde{e}_p$, $p = 0,\ldots,P$, whose $L^2(\Omega^0)$ norm are close to one: $\tilde{e}_p = \mu_p e_p$ and $|\mu_p|$ is close to one. More precisely, assume that there exists ϵ, $0 < \epsilon < 1$, such that, for all p, $0 \leqslant p \leqslant P$, $|\mu_p^2 - 1| \leqslant \epsilon$.

Consider a function $u \in H^1(\Omega^0)$ supported for example in Y^0. Call $\pi^P(u)$ the projection of u onto Λ^P:

$$\pi^P(u) = \sum_{p=0}^{P}(u, e_p)e_p = \sum_{p=0}^{P}\left(\int_{Y^0} ue_p\right)e_p.$$

The function $\pi^P(u)$ cannot be computed directly since e_p are not available. What can be computed is $\tilde{\pi}^P(u) = \sum_{p=0}^{P}(u,\tilde{e}_p)\tilde{e}_p = \sum_{p=0}^{P}(\int_{Y^0} u\tilde{e}_p)\tilde{e}_p$. It is clear that

$$\pi^P(u) - \tilde{\pi}^P(u) = \sum_{p=0}^{P}(1-\mu_p^2)(u, e_p)e_p = \sum_{p=0}^{P}(1-\mu_p^2)(\pi^P(u), e_p)e_p,$$

therefore $\|\pi^P(u) - \tilde{\pi}^P(u)\|_{L^2(\Omega^0)} \leqslant \epsilon\|u\|_{L^2(\Omega^0)}$. The numerical test below will confirm the fact that the method described in Section 8.2 permits to approximate correctly the projection of a compactly supported function on Λ^P.

8.4. Numerical computation of the spectrum

We use the domain and mesh displayed in Fig. 3 (Ω^0 is obtained by dilating the domain described in Section 2 by the factor π) and we wish to compute the lower part of the spectrum $Sp^{N,0}$, ($k < 40$), with the method proposed in Section 8.1. Before presenting the numerical results, we review known theoretical results on the density of eigenvalues for irregular domains.

8.4.1. The Weyl–Berry formula

Asymptotics of the density of states is an old problem which started with the well-known Weyl formula: if Ω is an open subset of $\mathbb{R}^d$ then $\aleph^D(k)$, the number of Dirichlet eigenvalues smaller than k behaves like

$$\aleph^D(k) \sim (2\pi)^{-d}\mathscr{B}_d|\Omega|_d k^{d/2} \quad \text{when } k \to \infty,$$

where $\mathscr{B}_d$ is the volume of the unit ball of dimension d, and $|\Omega|_d$ is the volume of the domain Ω.

In the present case, it can be seen that $|\Omega^0|_2 = 16\pi^2$ and that $\mathscr{B}_2 = \pi$. The Weyl formula is

$$\aleph^D(k) \sim 4\pi k \quad \text{as } k \to \infty. \tag{70}$$

When Ω has a smooth boundary (and under some extra conditions), a second term in the expansion of $\aleph^D(k)$ can be obtained: it is of the form $c_d k^{(d-1)/2}$, with a constant c_d depending on the length of the boundary (cf. [14]). When the boundary is irregular, the second term depends on the Minkowski dimension of the boundary. The Minkowski dimension is related to the volume of the ϵ-neighborhoods $\partial\Omega^\epsilon$ of the boundary $\partial\Omega$. More precisely, the Minkowski measurability can be defined relative to a gauge function $g : \mathbb{R}^+ \to \mathbb{R}^+$ increasing and with some extra properties (cf. [13], part 2). The boundary is said to be g-Minkowski measurable if

$$0 < \liminf_{\epsilon\to 0} \epsilon^{-d} g(\epsilon)|\partial\Omega^\epsilon \cap \Omega|_d = \limsup_{\epsilon\to 0} \epsilon^{-d} g(\epsilon)|\partial\Omega^\epsilon \cap \Omega|_d < \infty.$$

In this case, the remainder term $\aleph^D(k) - (2\pi)^{-d}\mathscr{B}_d|\Omega|_d k^{d/2}$ is proved to be of order (cf. [13], Theorem 2.12) $O(1/g(\frac{1}{\sqrt{k}}))$ (and it is expected to be comparable with this value).

In our case, it is easy to verify that $|(\partial\Omega^0)^\epsilon \cap \Omega^0|_2$ is equivalent to $\epsilon \log(1+\frac{1}{\epsilon})/\log 2$, which means that the boundary is g-Minkowski measurable, with $g(x) = \frac{x}{\log(1+\frac{1}{x})}$. Hence, the remainder term for the counting function of the Dirichlet eigenvalues $\aleph^D(k) - 4\pi k$ is of order $O(\sqrt{k}\log k)$.

Remark 11. Note that the Dirichlet problem mentioned here consists of imposing a Dirichlet condition on all $\partial\Omega^0$, so it is not problem (23). The same results hold for the Neumann boundary condition provided that an extra regularity condition (the "C' condition") is satisfied, see [13]. The "C' condition" is not satisfied by the domain Ω^0 under consideration, so the previous estimates are not known to be true for the Neumann problem.

Note also that Berger (see [5]) has studied the eigenvalue distribution and the asymptotics for the counting function for Dirichlet and Neumann problems in a domain with a snowflake boundary, which does not fulfill the conditions of [13]. The geometric construction in [5] differs from the present one, but it seems possible to adapt Berger's arguments; this remains to be done.

We are going to verify numerically the previously mentioned estimates for the Neumann problem.

8.4.2. The numerical computation of $Sp^{N,0}$

We have computed numerically the first part of the spectrum $Sp^{N,0}$ (i.e. $k < 40$) by the method proposed in Section 8.1. More precisely, for a chosen step Δk, we use the following method:

Algorithm 5. $k = k_{\min}$; compute T_k^0 (its discrete version) by the construction given in Section 5.3; compute the eigenvalues and the eigenvectors of T_k^0; while ($k < k_{\max}$)

- As long as the signature of T_k^0 does not change sign
  - $k = k + \Delta k$;
  - compute T_k^0 (its discrete version); compute the eigenvalues and the eigenvectors of T_k^0;
- the signature of T_k^0 has changed between k and $k - \Delta k$, we run a dichotomy method in order to compute a singular value k_{sing} between k and $k - \Delta k$. This may be either an eigenvalue (in $Sp^{N,0}$) of the Neumann problem, if $\ker(T^0_{k_{\text{sing}}}) \neq \{0\}$, or a value in $Sp^{D,0}$;
- set $k = \alpha k_{\text{sing}} + (1-\alpha)k$; ($\alpha$ is a fixed parameter, $0 < \alpha < 1$); compute T_k^0; compute the eigenvalues and the eigenvectors of T_k^0.

We took $\Delta k = 10^{-3}$ for $k \leqslant 10$ and $\Delta k = 10^{-2}$ for $10 < k \leqslant 40$. Of course, a drawback of this method is that it may miss an eigenvalue if there are more than one singular values between two successive test values of k.

In our C++ code, we used the functions contained in the library GSL (GNU Scientific Library) for computing the eigenvalues and eigenvectors of T_k^0.

Recall that if $u \neq 0$ belongs to the nullspace of T^0_k, then k (resp. $\mathscr{H}^0_k(u)$) is an eigenvalue (resp. eigenmode) of the Neumann problem. Thus, for a fixed integer $n \geqslant 0$, once k and u belonging to the nullspace of T^0_k is found, the restriction of the related eigenmode to Y^n can be found by using Algorithm 3.

8.4.3. The results

We numerically compute the counting function $\aleph$ of the Neumann eigenvalues. The "C' condition" is not satisfied, so the previous estimates are not known to be true for $\aleph$.

In Fig. 7, we have plotted $\aleph_h(k)$, the number of computed eigenvalues smaller than k vs. k, for $k < 40$, and the graph of $k \mapsto 4\pi k$. We see that the Weyl estimate (70) is very well satisfied by $\aleph_h(k)$. This indicates that the Weyl estimate is true for $\aleph$.

We go further and plot the remainder term $\aleph_h(k) - 4\pi k$. We have tried to fit this function by a function of the type $f(k) = (a\log(k) + b)\sqrt{k} + c$. The parameters a, b, c have been computing by using a least square algorithm in the interval $k = [0, 10]$. In Fig. 8, we plot the function $\aleph_h(k) - 4\pi k$ and $f(k)$, for $k \in (0, 30)$. Although the least square algorithm has been used to fit the function in the region $(0, 10)$, we see that $f(k)$ approaches $\aleph_h(k) - 4\pi k$ well, for $k \in (10, 30)$.

In Fig. 9, we have plotted the restrictions of the second and sixth eigenmodes (not normalized yet) to Y^2. These views respectively correspond to the contour lines at the top-left and middle-left of Fig. 10.

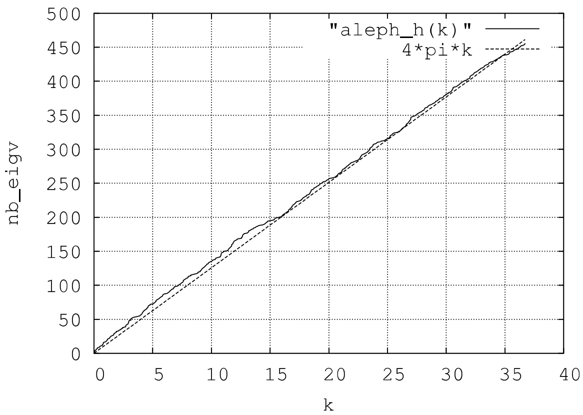


Fig. 7. The computed spectrum $Sp^{N,0}$: the number $\aleph_h(k)$ of eigenvalues smaller than k vs. k, for $k < 40$, and the graph of $k \mapsto 4\pi k$.

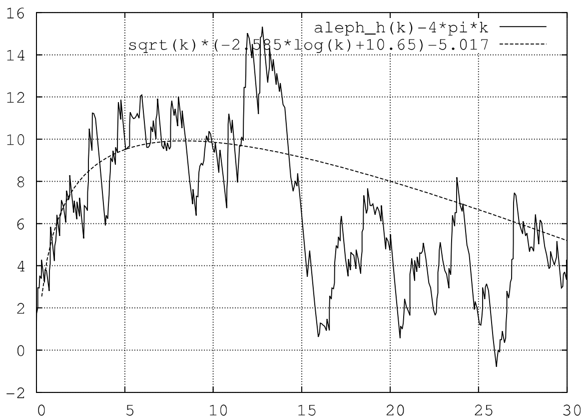


Fig. 8. The remainder $\aleph_h(k) - 4\pi k$ and $f(k)$.

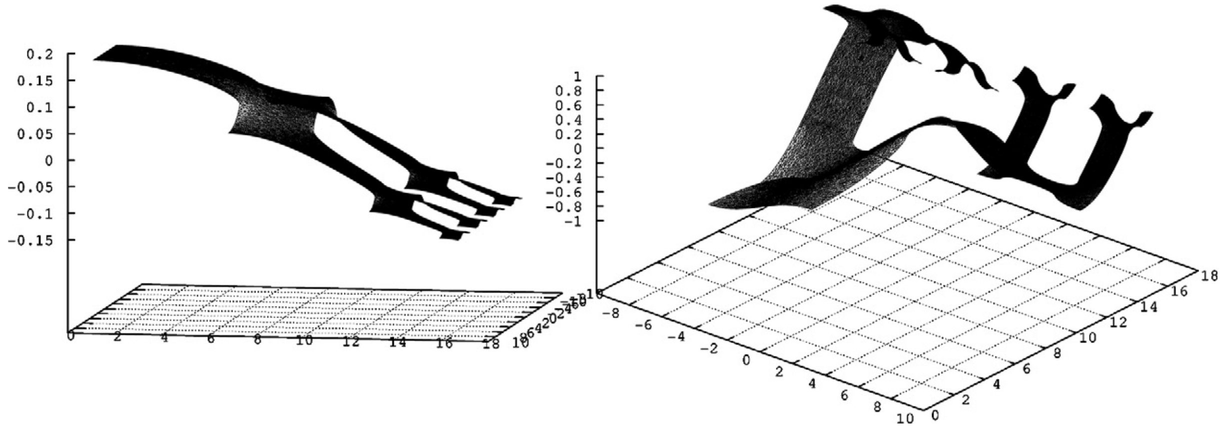


Fig. 9. The second (left side) and sixth (right side) eigenmodes, restricted to Y^2.

It is interesting to compare the spectrum found by the present method with the spectrum of the Neumann problem in Y^2, computed with a standard finite element method (the software FreeFem++ and the library Arpack for the generalized eigenvalue problem have been used). Table 1 contains the smallest eigenvalues of the two problems. The lower parts of the spectra clearly differ from each other; in particular, the second eigenvalue of the Neumann problem in Y^2 is smaller than the second eigenvalue of the Neumann problem in Ω^0. It is also possible to compare some of the eigenmodes: in Fig. 10, we see that the eigenmodes of the first problem, respectively associated with the eigenvalues 2.06×10^{-2}, 1.97×10^{-1} and 4.92×10^{-1}, somehow resemble the eigenmodes of the Neumann problem in Y^2 respectively associated with the eigenvalues 2.7×10^{-2}, 2.7×10^{-1} and 5.5×10^{-1}. Note that the increments in value from one contour to the next are not the same on the left and right sides of Fig. 10. For example, even if the top right/left figures seem very close to each other, the functions actually differ (the isovalues are closer to each other on the left, which is consistent with the fact that $2.06 \times 10^{-2} < 2.7 \times 10^{-2}$). It would be interesting (but too long for the present article) to make further numerical comparisons, in particular with the spectrum of the Neumann problem in Y^n, as n varies.

8.5. Modal decomposition of a compactly supported function and application to the numerical simulation of the wave equation in Ω^0

8.5.1. Modal decomposition of a compactly supported function

To test the normalization of the eigenmodes described in Section 8.2, we choose a compactly supported function u and we compare u with $\tilde{\pi}^P u$ for $1 \leqslant P \leqslant 300$. We choose

$$u = \left(\frac{x_1^2 - \pi^2}{\pi^2}\right)^4 \left(\frac{x_2(3\pi - x_2)}{\frac{9\pi^2}{4}}\right)^3 1_{-\pi<x_1<\pi} 1_{0<x_2<3\pi} e^{-x_1^2 - \left(x_2 - \frac{3\pi}{2}\right)^2}. \tag{71}$$

In the left part of Fig. 11, we plot the error $\|u - \tilde{\pi}^P u\|_{L^2(Y^0)}$ as a function of P. We see that the error decays as P tends to infinity. This shows that the family $(\tilde{e}_p)$ introduced in Section 8.2 is close to orthonormal and that the normalization of the eigenmodes by the perturbation method is accurate. In the right part of Fig. 11, we plot the reconstructed function $\tilde{\pi}^P u|_{Y^0}$ for $P = 300$. There is no visible difference between u and $\tilde{\pi}^{300}(u)$. More quantitatively, we obtain for $P = 300$, that $\|u - \tilde{\pi}^P u\|_{L^\infty(Y^0)} \leqslant 10^{-2}$.

To further test the normalization procedure, we compute $\alpha_p = \int_{Y^0} (u - \tilde{\pi}^{300} u)\tilde{e}_p$ for $0 \leqslant p \leqslant 300$; if $\tilde{e}_p$ matched e_p for all p, $0 \leqslant p \leqslant 300$, then the numbers α_p would be exactly 0. In Fig. 12, we plot α_p as a function of p. We see that these numbers never exceed 3×10^{-3}, which confirms the fact that the family $\tilde{e}_p$ is very close to being orthonormal.

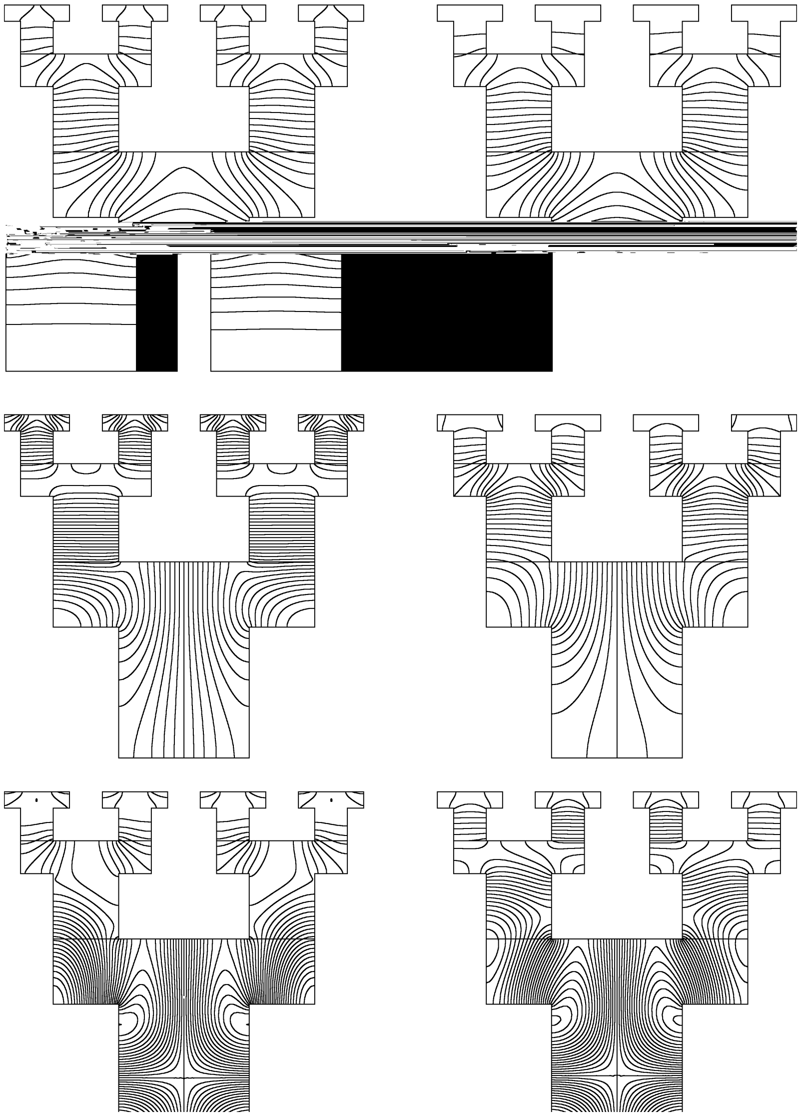


Fig. 10. Left: the restrictions to Y^2 of the eigenmodes of the Neumann problem in Ω^0, corresponding with the eigenvalues 2.06×10^{-2}, 1.97×10^{-1} and 4.92×10^{-1}. Right: the eigenmodes of the Neumann problem in Y^2, corresponding with the eigenvalues 2.7×10^{-2}, 2.7×10^{-1} and 5.5×10^{-1}. The contours on the left and right figures do not correspond to the same values.

Table 1
The smallest eigenvalues of the Neumann problem in Ω^0 (computed by the method in Section 8.4.2) and in Y^2 (computed with a standard finite element method)

Ω^0	0	2.06×10^{-2}	7.84×10^{-2}	8.24×10^{-2}	1.63×10^{-1}	1.97×10^{-1}	2.87×10^{-1}
Y^2	0	1.3×10^{-2}	2.7×10^{-2}	8.6×10^{-2}	9.6×10^{-2}	1.1×10^{-1}	2.3×10^{-1}

8.5.2. Wave propagation in Ω^0

Finally, it is possible to use the above modal decomposition of a compactly supported function for solving a Cauchy problem for the wave equation in Ω^0 with compactly supported initial data: we wish to compute as accurately as possible the restriction to Y^n (here, we shall restrict ourselves to $n = 0$) of the solution to the following problem:

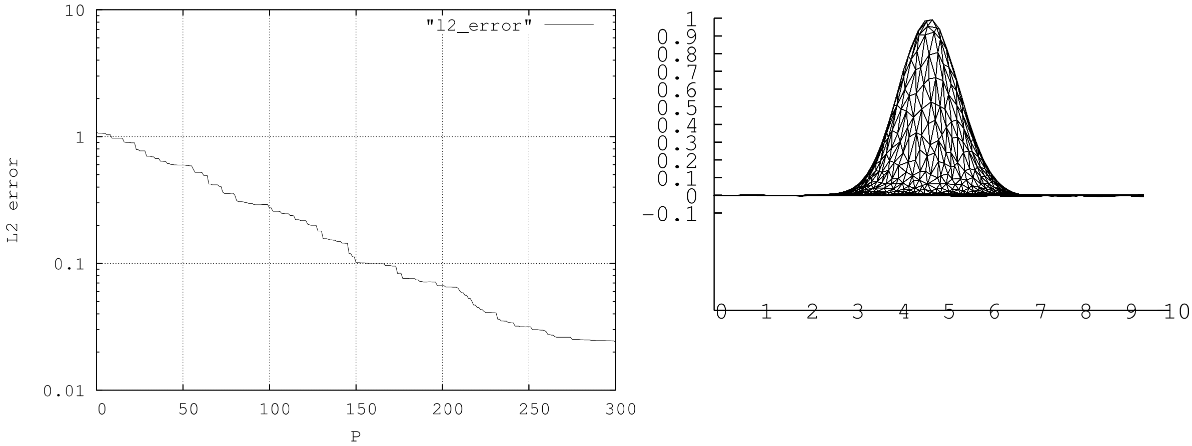
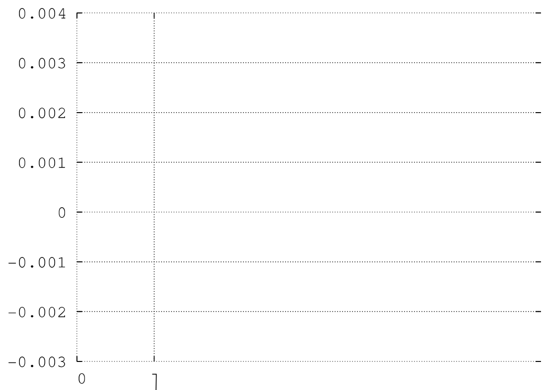


Fig. 11. Left: the error $\|u - \tilde{\pi}^P u\|_{L^2(Y^0)}$ vs. P. Right: the function $\tilde{\pi}^{300} u|_{Y^0}$ (viewed from the east).



$$
\begin{aligned}
&\frac{\partial^2 w}{\partial t^2} - \Delta w = 0 \quad \text{in } (0,T) \times \Omega^0,\\
&\frac{\partial w}{\partial n} = 0 \quad \text{on } (0,T) \times (\Gamma^0 \cup \Sigma^0),\\
&w|_{t=0} = u_0 \quad \text{in } \Omega^0,\\
&\frac{\partial w}{\partial t}|_{t=0} = u_1 \quad \text{in } \Omega^0,
\end{aligned}
\tag{72}
$$

where u_0 and u_1 are two functions supported for example in Y^0. It is possible to approximate the modal decompositions of u_0 and u_1, namely to compute $\tilde{\pi}^P u_0 = \sum_{p=0}^{P} \beta_p \tilde{e}_p$ and $\tilde{\pi}^P u_1 = \sum_{p=0}^{P} \gamma_p \tilde{e}_p$; then, one may solve a Cauchy problem close to (72)

$$
\begin{aligned}
&\frac{\partial^2 \tilde{w}}{\partial t^2} - \Delta \tilde{w} = 0 \quad \text{in } (0,T) \times \Omega^0,\\
&\frac{\partial \tilde{w}}{\partial n} = 0 \quad \text{on } (0,T) \times (\Gamma^0 \cup \Sigma^0),\\
&\tilde{w}|_{t=0} = \tilde{\pi}^P u_0 \quad \text{in } \Omega^0,\\
&\frac{\partial \tilde{w}}{\partial t}|_{t=0} = \tilde{\pi}^P u_1 \quad \text{in } \Omega^0,
\end{aligned}
\tag{73}
$$

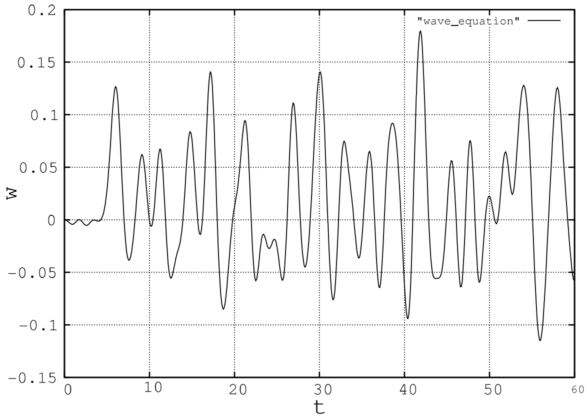


Fig. 13. $\tilde{w}(t,a)$ vs. t, where $\tilde{w}$ is the solution to (73) with u_0 given by (71), and $u_1 = 0$.

because

$$\tilde{w}(t,x) = (\beta_0 + \gamma_0 t)\tilde{e}_0 + \sum_{p=1}^{P}\left(\beta_p \cos(\sqrt{k_p}t) + \frac{\gamma_p}{\sqrt{k_p}}\sin(\sqrt{k_p}t)\right)\tilde{e}_p(x).$$

We stress the fact that $\tilde{e}_0$ is a constant. In Fig. 13, we have plotted the value of $\tilde{w}(t,a)$ as a function of time, for $a = \left(\frac{3\pi}{2}, 3\pi\right)$, for $u_0 = u$ given by (71), and $u_1 = 0$.

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